Note Title 9/13/2007

Lecture 6

Parameterization / specifying Rotations

$[r_{ij}]$

Rot. 3x3 matrices, orthonormal

9 elements

$R(a, b, c)$

1. Col. vec. are unit vec. $\leftarrow 3$
2. " " " mutually orth. $\leftarrow 3$
3. det(R) = 1 || $\leftarrow$ does not change the # of ind. par.

$\Rightarrow$ 3 ind. elements in the matrix

Several diff. conventions / ways to rep. these ind. elements.

1) Fixed angle representation (axes) $\{A\}$

1) around $\hat{X}_A$ by angle γ

2) " $\hat{Y}_A$ " " β

3) " $\hat{Z}_A$ " " α

pre mult

pre mult

$$R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha)\, R_Y(\beta)\, R_X(\gamma)$$

Given γ, β, α, we can compute $R_{XYZ}(\gamma, \beta, \alpha)$

We computed $R_Z(\theta)$ in Lecture 2.

$$= \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

similarly, $R_Y(\theta)$, $R_X(\theta)$ can be computed. see below.

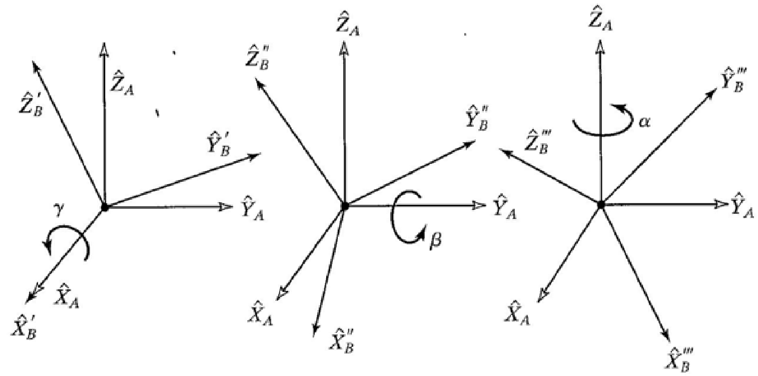

FIGURE 2.17: X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

The derivation of the equivalent rotation matrix, ${}^A_B R_{XYZ}(\gamma, \beta, \alpha)$, is straightforward, because all rotations occur about axes of the reference frame; that is,

$$
\begin{aligned}
{}^A_B R_{XYZ}(\gamma, \beta, \alpha) &= R_Z(\alpha) R_Y(\beta) R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}, \qquad (2.63)
\end{aligned}
$$

where $c\alpha$ is shorthand for $\cos\alpha$, $s\alpha$ for $\sin\alpha$, and so on. It is extremely important to understand the order of rotations used in (2.63). Thinking in terms of rotations as operators, we have applied the rotations (from the *right*) of $R_X(\gamma)$, then $R_Y(\beta)$, and then $R_Z(\alpha)$. Multiplying (2.63) out, we obtain

$$
{}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. \qquad (2.64)
$$

Keep in mind that the definition given here specifies the order of the three rotations.

2) Euler Angle rep. : Successive rotations are carried out with respect to "current" frame and not orig frame.

$$\{A\} \xrightarrow[\hat{Z}_A,\ \alpha]{} \{B'\} \xrightarrow[\hat{Y}_{B'},\ \beta]{} \{B''\} \xrightarrow[\hat{X}_{B''},\ \gamma]{} \{B'''\}$$

$$R_{Z'Y'X'}(\alpha, \beta, \gamma) = {}^{A}_{B'}R \; {}^{B'}_{B''}R \; {}^{B''}_{B'''}R$$

$$= R_Z(\alpha)\, R_Y(\beta)\, R_X(\gamma)$$

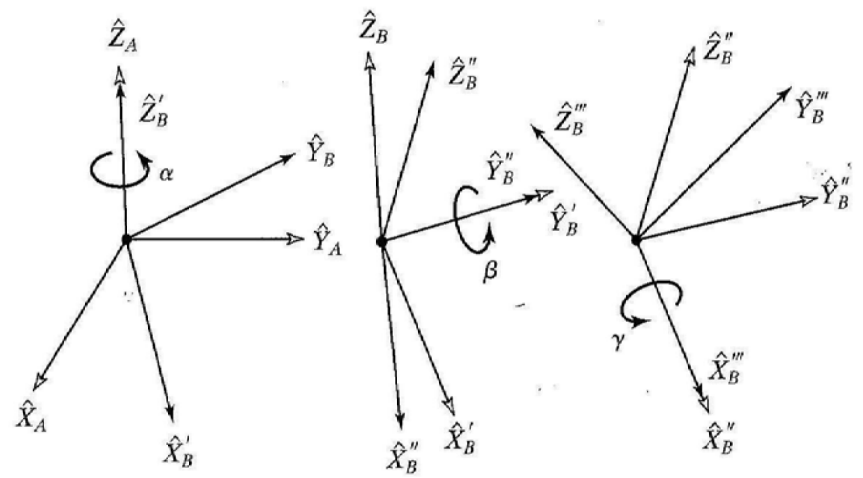

FIGURE 2.18: Z–Y–X Euler angles.

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frame.
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Z–Y–Z

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St
ab
Z_b

Rc
this is ar

Current axes rotation $\rightarrow$ post. mult. of matrices

fixed " " $\rightarrow$ pre mult of matrices.

why? the connection? later?

forward comp. $\xrightarrow{\text{easy}}$ Given α, β, γ, compute $R_{XYZ}(\gamma, \beta, \alpha)$

$$\begin{tikzcd} \alpha, \beta, \gamma \arrow[r, shift left, "f."] & R \arrow[l, shift left, bend left, "\text{inv.}"] \end{tikzcd}$$

inverse " $\longrightarrow$ Given, R, compute α, β, γ.

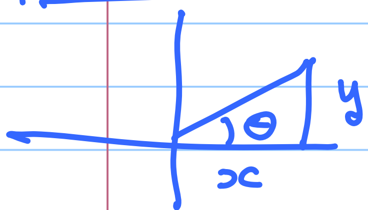
Given $R_{z'y'x'}(\alpha, \beta, \gamma)$ = see Eqn. 2.71 in text, compute $\underline{\alpha, \beta, \gamma}$

$\underline{\text{Given}}$ $\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$, solve for α, β, γ

Comparing element by element, you get 9 eqns.
highly non-linear, multiple variables (α, β, γ).
Trick: look for eqns. from where you
can eliminate some variables using
trig. identities.

Always use Atan2(y, x), it resolves the angle
uniquely in all four quadrants.

$$\boxed{\mathrm{Atan2}(y,x)=\theta}$$



by inspection,

$$\left.\begin{aligned} r_{11} &= c\alpha\, c\beta \\ r_{21} &= s\alpha\, c\beta \end{aligned}\right\} \Rightarrow c\beta = \pm\sqrt{r_{11}^2 + r_{12}^2} \quad \text{①}$$

Now,

$$r_{31} = -s\beta \quad \text{②}$$

Using ① + ②, and assuming $|\beta| \leq 90^\circ$,

+ve sign

we get $\boxed{\beta} = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{12}^2}\right)$

Now, we know β. Going back to r_{11} & r_{21}, we can divide by $c\beta$, to get $c\alpha$, $s\alpha$, respectively.

It follows,

$$\boxed{\alpha} = \text{Atan2}\left(r_{21}/c\beta,\ r_{11}/c\beta\right)$$

assumes $\beta \neq 90^\circ$. More on it later.

by inspection,

$$\left.\begin{aligned} r_{32} &= c\beta\, s\gamma \\ r_{33} &= c\beta\, c\gamma \end{aligned}\right\} \Rightarrow \boxed{\gamma} = \text{Atan2}\left(r_{32}/c\beta,\ r_{33}/c\beta\right).$$

Done: we have solved for α, β, γ. assuming $\beta \neq 90^\circ$

<u>What if $\beta = 90^\circ$</u> : matrix from becomes

$$\begin{bmatrix} 0 & s(\gamma-\alpha) & c(\gamma-\alpha) \\ 0 & c(\gamma-\alpha) & -s(\gamma-\alpha) \\ -1 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ we can only solve for $(\gamma-\alpha)$!!

and not γ, α individually. Also

$\Rightarrow$ infinite solns. all γ, α such that $\gamma-\alpha$ remains same!

such situations are called "singularities" and need to be explicitly taken into account.

why do they happen: often axes line up!!

$\beta = 90^\circ \Rightarrow \hat{X}'''$ (axis of third rotation, by γ) lines up with $-\hat{Z}'$!! $= -\hat{Z}$

so net rot. of $(\gamma - \alpha)$ around $-\hat{Z}$ axis.

3) Equivalent angle - axis rep.

$\underline{\underline{R_k(\theta)}}$

given axis of rot. is a unit vec. $\hat{k}$

$\hat{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$

θ = amount of rotation.

$|\hat{k}| = 1$

<u>form of this matrix : ??</u>

basic idea: Carry out successive rotations to align $\hat{k}$ with $\hat{z}$ axis (or any principal axis), Carry out rot. by θ around $\hat{z}$, then rotate $\hat{z}$ back to $\hat{k}$.

Convince yourself (Carry this out) that it physically leads to same final orientation !!

see figure below : Note in fig :

$$k_x = r_x$$

$$k_y = r_y$$

$$k_z = r_z$$

$$\theta = \phi$$

$$\sin\alpha = \frac{r_y}{\sqrt{r_y^2+r_z^2}} \qquad \cos\alpha = \frac{r_z}{\sqrt{r_y^2+r_z^2}}$$

$$\sin\beta = r_x \qquad \cos\beta = \sqrt{r_y^2+r_z^2}$$

Substituting into the above equation,

$$\mathbf{R}_{r,\phi} = \begin{bmatrix} r_x^2V\phi+C\phi & r_xr_yV\phi-r_zS\phi & r_xr_zV\phi+r_yS\phi \\ r_xr_yV\phi+r_zS\phi & r_y^2V\phi+C\phi & r_yr_zV\phi-r_xS\phi \\ r_xr_zV\phi-r_yS\phi & r_yr_zV\phi+r_xS\phi & r_z^2V\phi+C\phi \end{bmatrix}$$

$r = \hat{k}$
$\phi = \theta$

$C\phi = \cos\phi$
$V\phi = 1-\cos\phi$

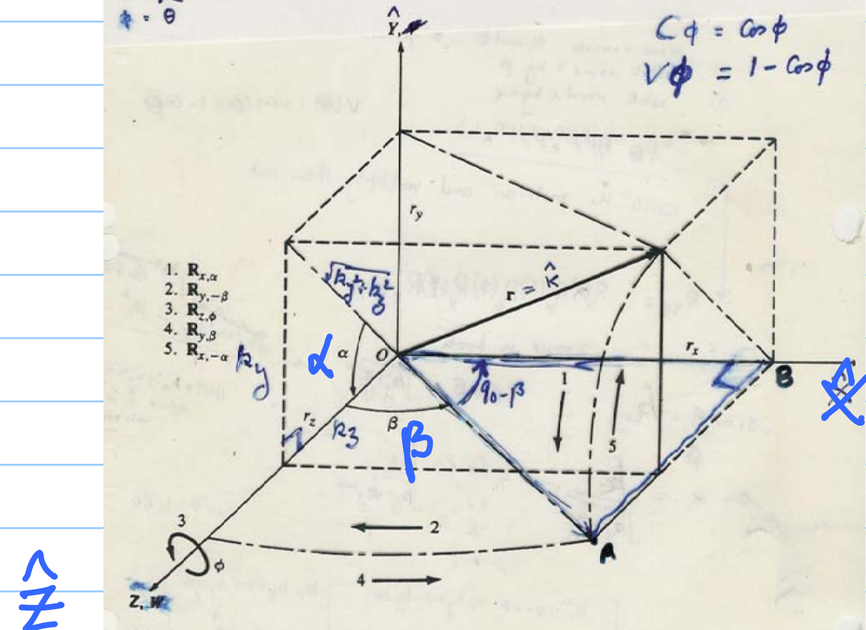


Figure 2.4 Rotation about an arbitrary axis.

Convince yourself that $R_{\hat{k}}(\theta)$ is equivalent to following rotations:

1) Rotate $\hat{k}$ by α around $\hat{x}$ axis. This brings $\hat{k}$ to $x-z$ plane.

2) Rotate by $-\beta$ around $\hat{y}$ axis. This brings $\hat{k}$ to align with $\hat{z}$.

actual θ rotation

3) Rotate by θ around $\hat{z}$.

4) Reverse of 2 : rotate by β around $\hat{y}$-axis.

5) Reverse of 1 : rotate by $-\alpha$ around $\hat{x}$.

Since all rot. are around fixed axes of the same frame, pre-multiply. So, we get:

$$R_{\hat{k}}(\theta) = R_{\hat{x}}(-\alpha)\, R_{\hat{y}}(\beta)\, R_{\hat{z}}(\theta)\, R_{\hat{y}}(-\beta)\, R_{\hat{x}}(\alpha)$$

α, β, are related to k_x, k_y as shown in Fig. by straightforward geometry. Multiplying all out, we get the result stated in text in Eqn. 2.80.