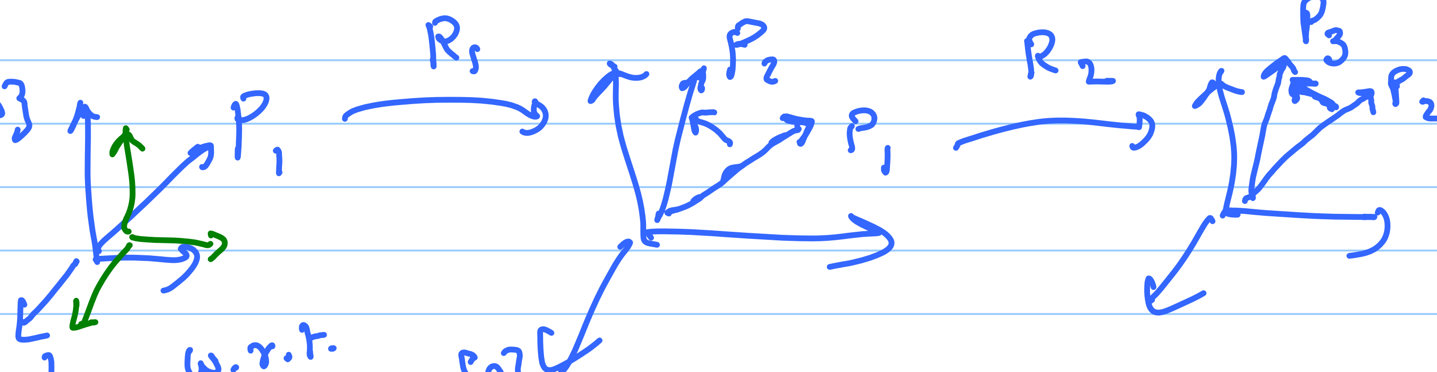
Lecture - 7

We have seen earlier, $\quad R\,{}^{A}P_1 = {}^{A}P_2$

1. fixed angle $\{A\}$

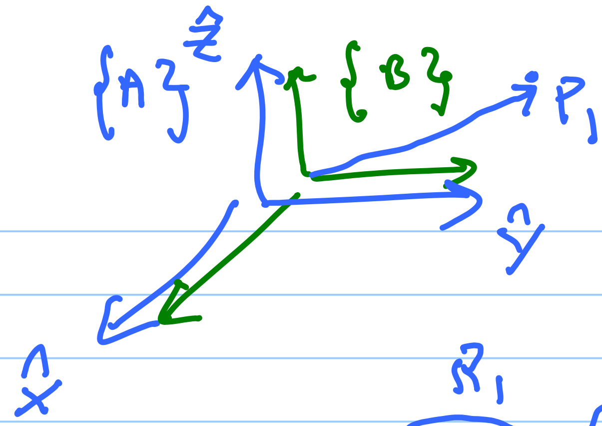


$$\left.\begin{array}{l} R_1 = R_x(\phi) \\ R_2 = R_y(\psi) \end{array}\right\} \Leftarrow \text{w.r.t. fixed frame } \{A\}$$

$$P_3 = R_2 R_1 P_1 \leftarrow \text{fixed axis rep}$$

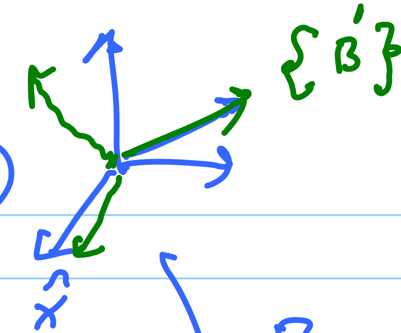
$$= R_y(\psi) R_x(\phi) \leftarrow$$

2. Euler Rep:

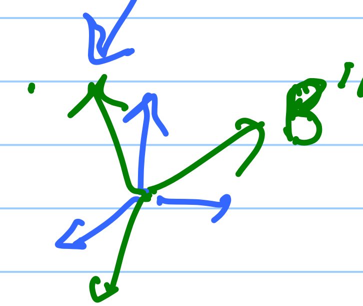


$R_1 = R_{\hat{X}}(\phi)$

$\hat{X}_R$



$R_2 = R_{\hat{y}_{B'}}(\psi)$



$${}^{A}_{B''}R = \underbrace{{}^{A}_{B'}R}_{R_1} \; \underbrace{{}^{B'}_{B''}R}_{R_2}$$

$$= R_x(\phi) \cdot \underline{\underline{R_y(\psi)}}$$

w.r.t. Current frame

$$\begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & s\phi \end{pmatrix} = R_y(\psi) \quad \text{w.r.t. current frame.}$$

$\Rightarrow$

Rule 1) Successive Rot. expressed w.r.t. fixed frame $\Rightarrow$ pre-mult. of corr. matrices

Rule 2) Successive Rot. expressed w.r.t. current frame $\Rightarrow$ post. mult. with corr. matrices

This has a deeper interpretation.

Rule 1 follows directly from composition of "transformations": $P_2 = R_1 P_1 \quad P_3 = R_2 P_2$

$\Rightarrow P_3 = R_2 R_1 P_1$

We will now apply rule (1) to Case 2 where second Rotation is carried out around $\hat{Y}_{B'}$. What is the matrix rep. of this rotation expressed in fixed frame?

In $R_{\hat{K}}(\theta)$ notation, this could be written as $R_{{}^{A}\hat{Y}_{B'}}(\psi)$, i.e. $\hat{Y}_{B'}$ expressed in $\{A\}$. We can also write this as ~~t~~three rotations carried out as follows:

showed via physical example using "Tinker Toy" frame

1) Rotat $\{B'\}$ back to $\{A\}$, i.e. carry out $R_{\hat{x}}^{-1}(\phi)$. This coincides $\{B\}$ with $\{A\}$.

2) Carry out $R_{\hat{y}}(\psi)$

3) Rotat $\{A\}$ back to $\{B'\}$, i.e. carry out $R_{\hat{x}}(\phi)$

$$\Rightarrow \underbrace{R_{\hat{x}}(\phi)\, R_{\hat{y}}(\psi)\, R_{\hat{x}}^{-1}(\phi)}_{\text{matrix rep. of rotation (carried out around } R_{\hat{y}_{B'}}\text{)}}$$

expressed w.r.t. $\{A\}$, the fixed frame.

So: total composed rotation

$$= \underbrace{R_{\hat{x}}(\phi)\, R_{\hat{y}}(\psi)\, \cancel{R_{\hat{x}}^{-1}(\phi)}}_{\text{2nd. rot.}}\ \underbrace{\cancel{R_{\hat{x}}(\phi)}}_{\text{first rot.}}$$

both expressed w.r.t. fixed frame $\{A\}$.

$$= R_{\hat{x}}(\phi)\, R_{\hat{y}}(\psi)$$

expression Called similarity Xform: expresses same "operation/transformation" but

w.r.t. diffrent co-ordinate frame.

Concept is general and can be applied to any linear Xformation and not just rotations.

We can also interpret it algebraically

Suppose we are rot. P_1 to P_2 w.r.t. $\hat{Y}_B$ axis by angle ψ. We can "view" the same operation w.r.t. a frame $\{A\}$ when ${}^A_B R$ is given.

So,

Now $\quad {}^{B}P_1 = \underbrace{{}^{B}_{A}R \; {}^{A}P_1}_{\text{co-ord xform}}$

Apply, $R_{\hat{y}_B}(4)$ to ${}^{B}P_1$:

$${}^{B}P_2 = R_{\hat{y}}(4) \cdot {}^{B}P_1 = R_{\hat{y}}(4)\, {}^{B}_{A}R \, {}^{A}P_1$$

Now, $\quad {}^{A}P_2 = {}^{A}_{B}R \; {}^{B}P_2 = \boxed{{}^{A}_{B}R \; R_{\hat{y}}(4) \; {}^{B}_{A}R} \; {}^{A}P_1$

Hence, we can write:

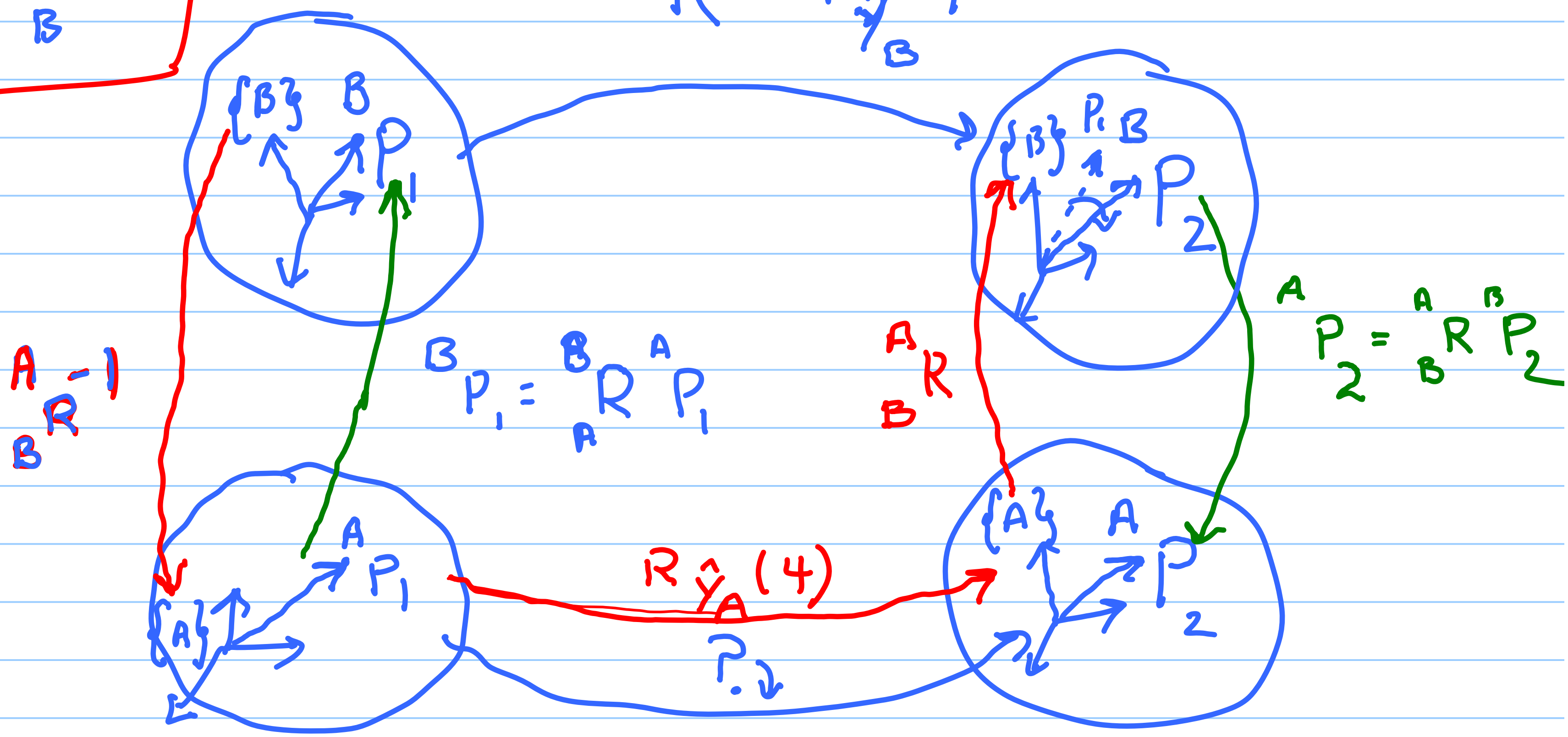
$$\Leftrightarrow \boxed{\left[{}^{A}_{B}R \; R_{\hat{Y}}(\psi) \left({}^{A}_{B}R \right)^{-1} \right]} {}^{A}P_1 = {}^{A}P_2$$

matrix rep. of rot. around $\hat{Y}_B$ expressed w.r.t. $\{A\}$, the fixed frame.

$\boxed{{}^{A}_{B}R \quad R \quad {}^{A}_{B}R^{-1}}$

This is captured in the foll. diag.

$$R = R_{\hat{y}_B}(4)$$



Please note that there are two ways of interpreting the above diag:

1) Rotating co-ordinate frames, and rotations w.r.t. fixed frames shown in red arrows

2) Xforming vectors from one frame to another green arrows.

NOTE: In RESPONSE to a question in class, I have modified the notes after the lecture to reflect the two interpretations.

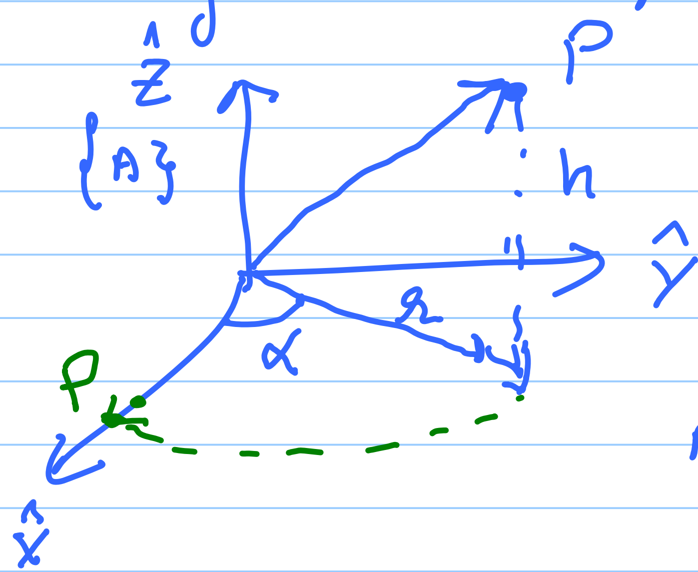
Position : 3x1 vect.

① Cartesian $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

② Cylindrical

(r, α, h)

$^{A}P = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$



$\begin{pmatrix} r\,c\alpha \\ r\,s\alpha \\ h \end{pmatrix}$

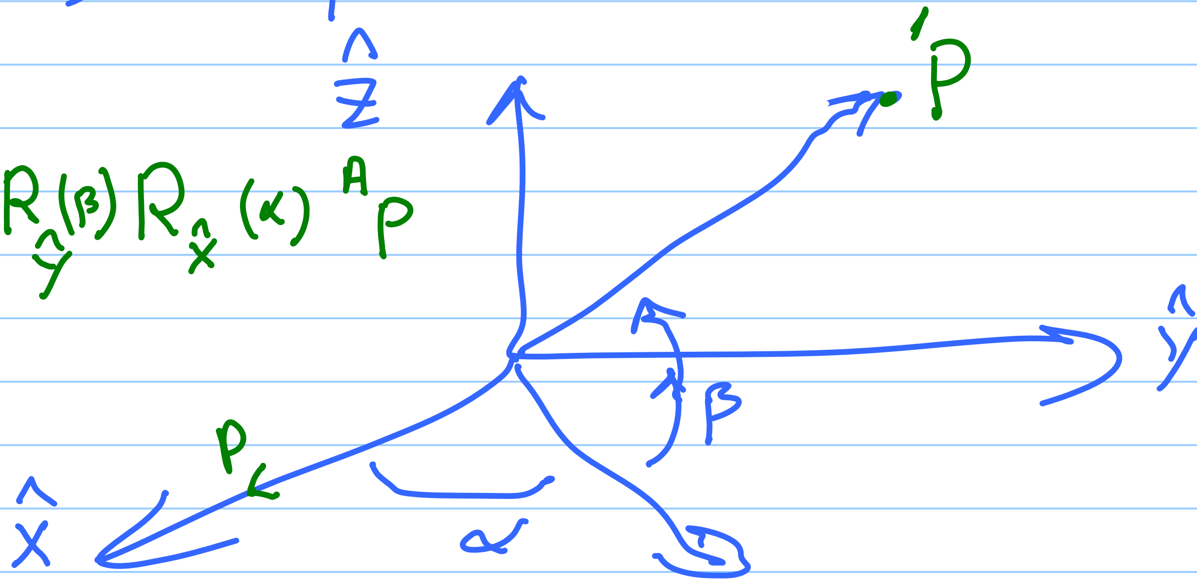
$^{A}P' = D_{Z}(h)\, R_{\hat{Z}}(\alpha)\, ^{A}P$

Check notation

$$q = \sqrt{q_x^2 + q_y^2 + q_z^2} \qquad D_Q(q) = \left(\begin{array}{c|c} \underline{0} & \begin{matrix} q_x \\ q_y \\ q_z \end{matrix} \\ \hline \underline{0} & 1 \end{array}\right) \left(\begin{array}{c|c} R_z(\alpha) & \underline{0} \\ \hline \underline{0} & 1 \end{array}\right)$$

3) spherical (See 2.18 prob)

$${}^{A}P' = R_{\hat{y}}(\beta) R_{\hat{x}}(\alpha)\, {}^{A}P \qquad {}^{A}P = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$



$$\Rightarrow \quad {}^{A}p' = \begin{pmatrix} r \cdot c\alpha \cdot c\beta \\ r \cdot s\alpha \cdot c\beta \\ r \cdot s\beta \end{pmatrix}$$

Summary: Math. of Rigid Body Motions

4x4 matrix transform

1) Trans.

2) Rotation

$$T = \left(\begin{array}{c|c} R & p \\ \hline \underline{0} & 1 \end{array}\right)$$

Three interpretations:

1) ${}^{A}_{B}T$: rep. $\{B\}$ w.r.t. $\{A\}$

2) Mapping ${}^{A}P = {}^{A}_{B}T \, {}^{B}P$

3) Operator: Motion

$$P_2 = T \, P_1$$

Use of vectors in mechanics

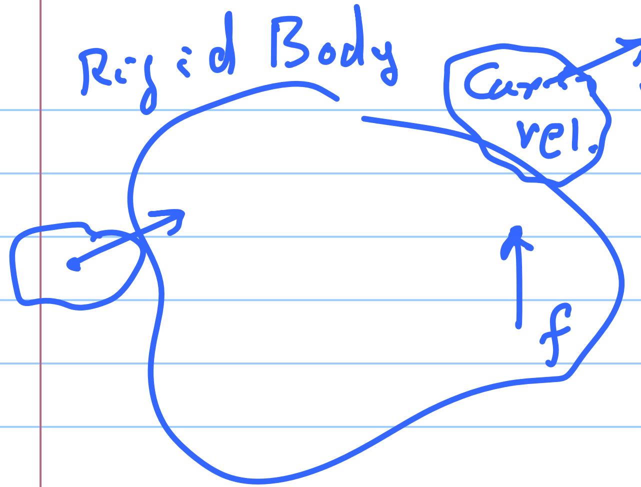
various entities: pos., vel., force ...

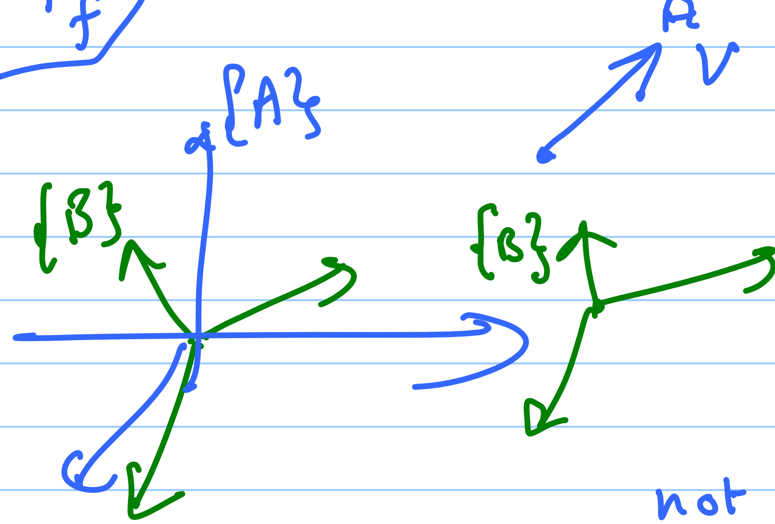
are rep. by vectors

Two types of vectors in mechanics:

① line-vector : force

② free-vector : velocity





$$^{A}V = {}^{A}_{B}R\,{}^{B}V$$

Position does not play a role in co-ordinate xformation of these vectors.