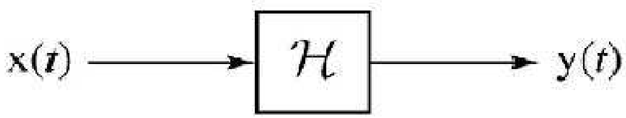
ENSC380
Lecture 11

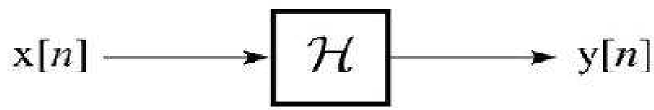
Objectives:

- Learn how to represent a periodic signal with its continuous time Fourier series (CTFS)

CT Impulse Response

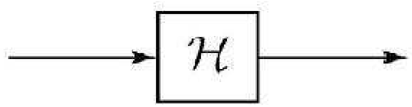
- In the previous lectures we learned that the response of LTI systems to complex exponentials (e^{st} or $e^{j\omega t}$, is

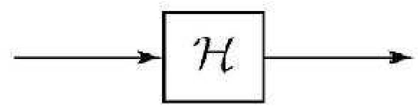




- Furthermore, if the input to an LTI system is a sinusoidal of frequency f, the output is

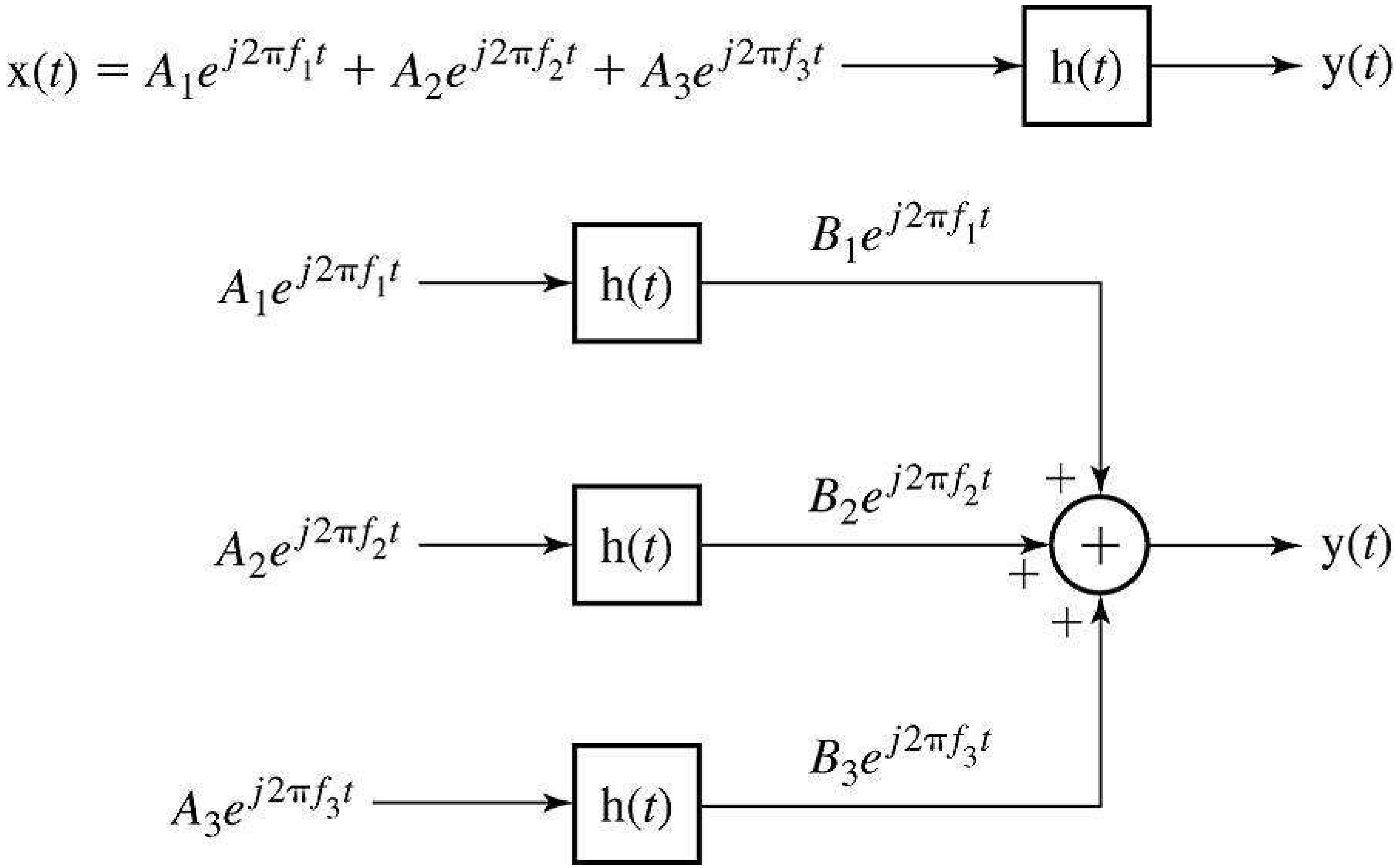
- Finally, if an excitation signal ($x(t)$ or $x[n]$) is a linear combination of complex exponentials (or real $\sin(.)$ or $\cos(.)$ signals), the response of the system is:





- Note: Complex exponentials are also called complex sinusoids because:

Complex Exponential Response



Fourier Series

- In early 19th century Jean Baptiste Joseph Fourier showed that signals may in fact be represented as linear combinations of complex exponentials (or real $\cos(.)$ and $\sin(.)$)

- He showed that **periodic** signals can be exactly represented as linear combinations of sinusoids.
- He also showed that for **aperiodic** signals, any **time limited portion** of them (e.g. from t_0 to $t_0 + T_p$) can be represented as a linear combination of complex exponentials.
- This linear combination is called **"Fourier Series"** (FS), for obvious reasons!
- We first consider continuous-time Fourier series (CTFS).

Illustration for Periodic

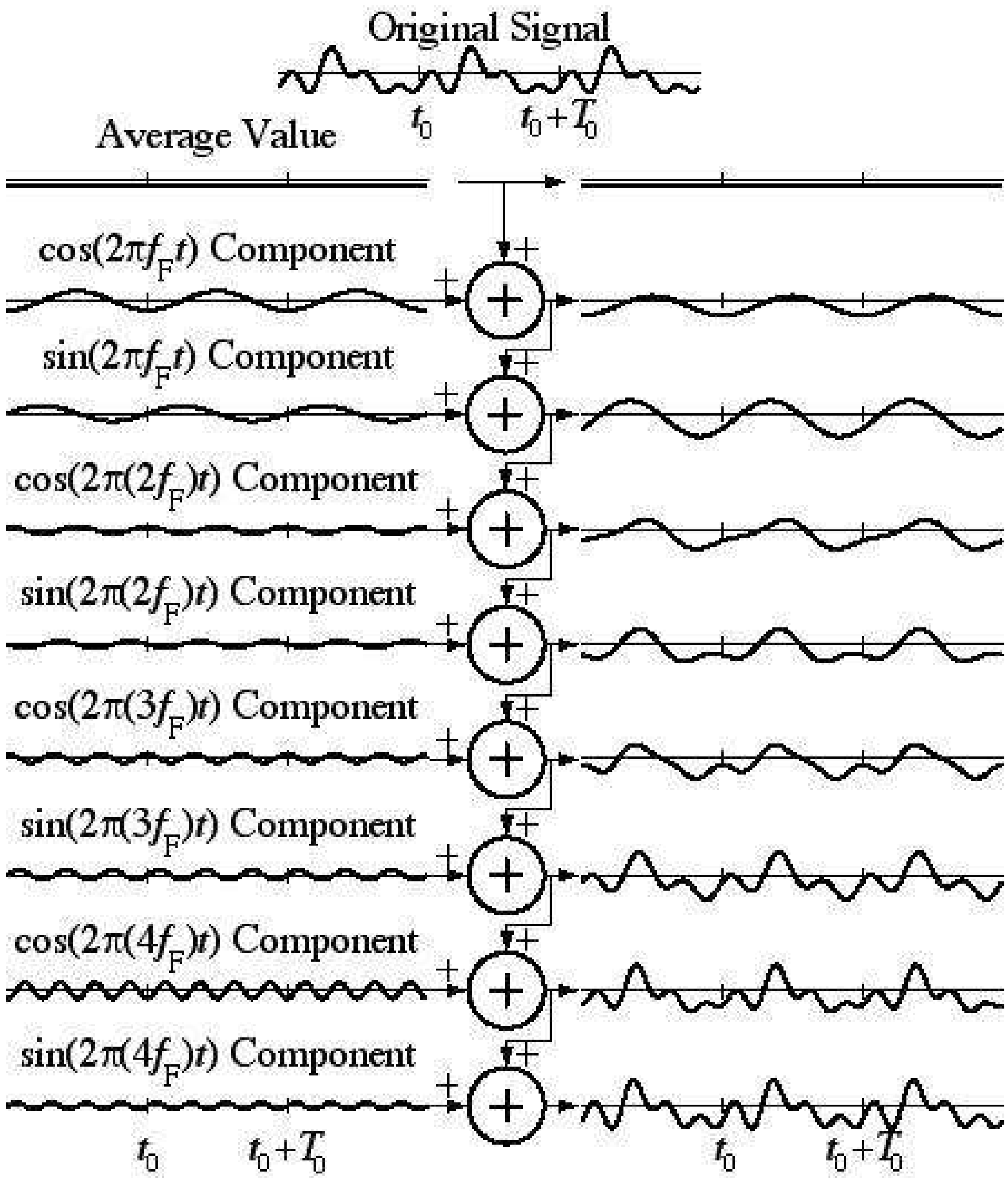
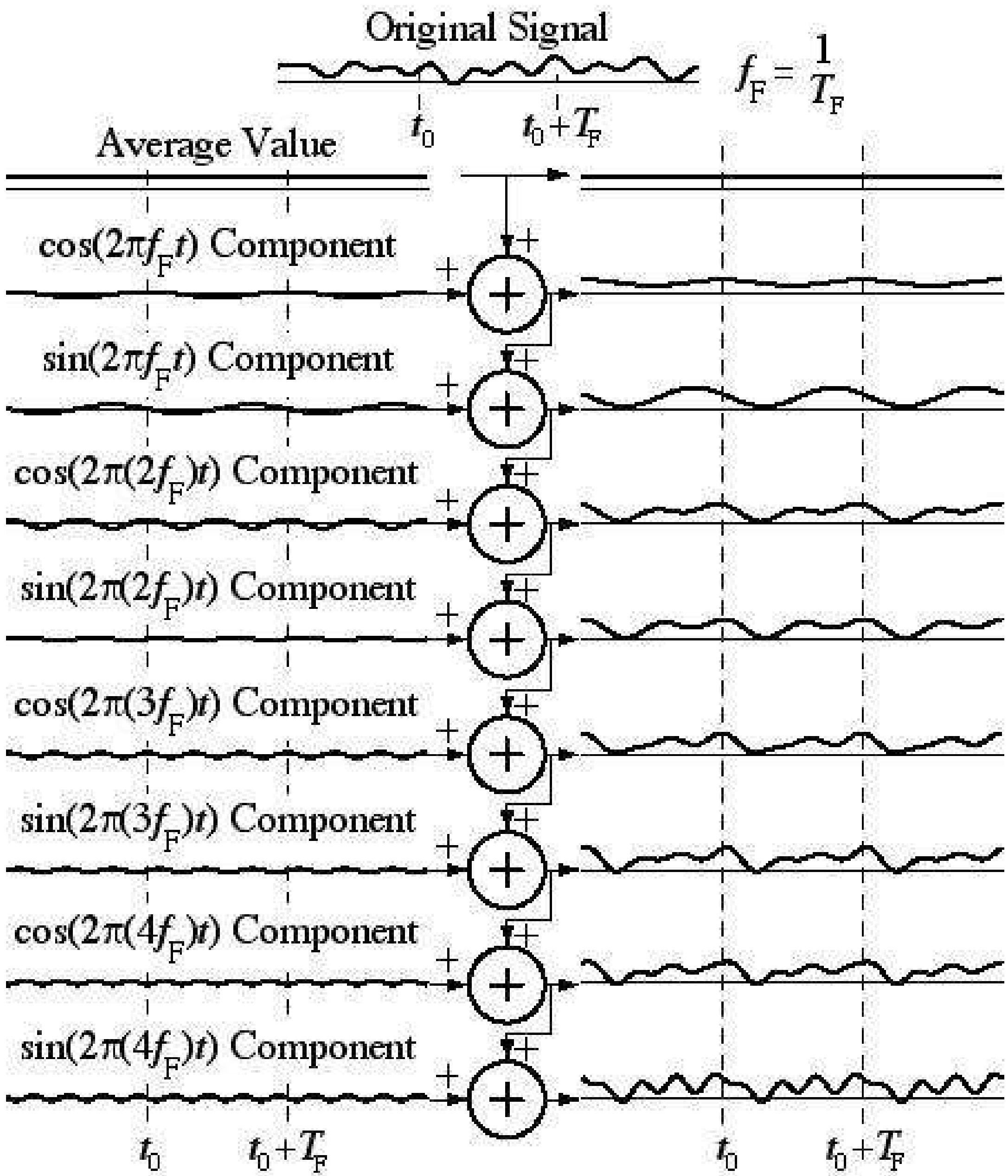


Illustration for Aperiodic



CTFS for Periodic Signals

- Consider a periodic signal, $x(t)$, with fundamental period T_0 (fundamental frequency $f_0 =$)
- The Fourier series representation for $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi(kf_0)t}$$

- In the above definition, f_0 is called the **fundamental frequency** of the Fourier series representation and kf_0 is the kth **harmonic** of the fundamental frequency, k is the **harmonic number**, and $X[k]$ is the k**th coefficient** of the FS. $X[k]$ is also referred to as the **harmonic function**.
- The question now is whether it is always possible to find the FS coefficients, $X[k]$, and if yes how?

FS coefficients

$x(t)$ periodic with period T_0, $f_0 = 1/T_0$:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi(kf_0)t}$$

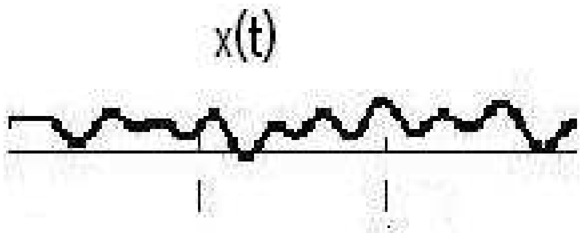
- To find $X[k]$ for $k = q$, (q integer), multiply both sides of the above equation by $e^{-j2\pi(qf_0)t}$ and integrate over one period ($t \in [t_0, t_0 + T_0]$).
- This will result in:

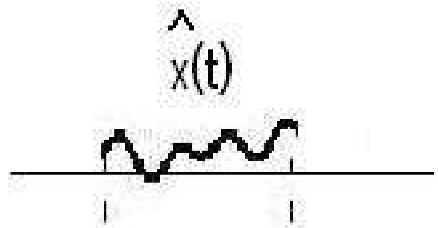
$$X[q] = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-j2\pi(qf_0)t} dt$$

- If the above integral converges to a finite value for all q, the FS coefficients can be found for all q, and the FS representation is valid.
- If the above integral does not converge, the signal does not have a FS representation.

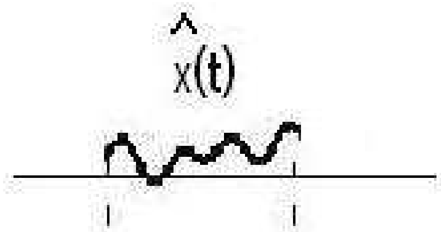
CTFS for Aperiodic Signals

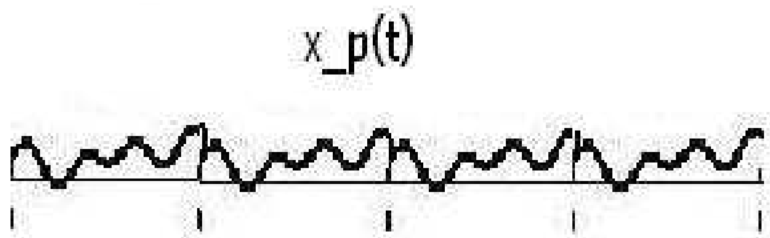
- Consider an aperiodic signal, $x(t)$. Extract a portion of $x(t)$ over a time interval $t_0 \leq t \leq t_0 + T_f$ (T_f arbitrary) and call it $\hat{x}(t)$:





- Create a periodic signal ($x_p(t)$) by repeating $\hat{x}(t)$ every T_f seconds:



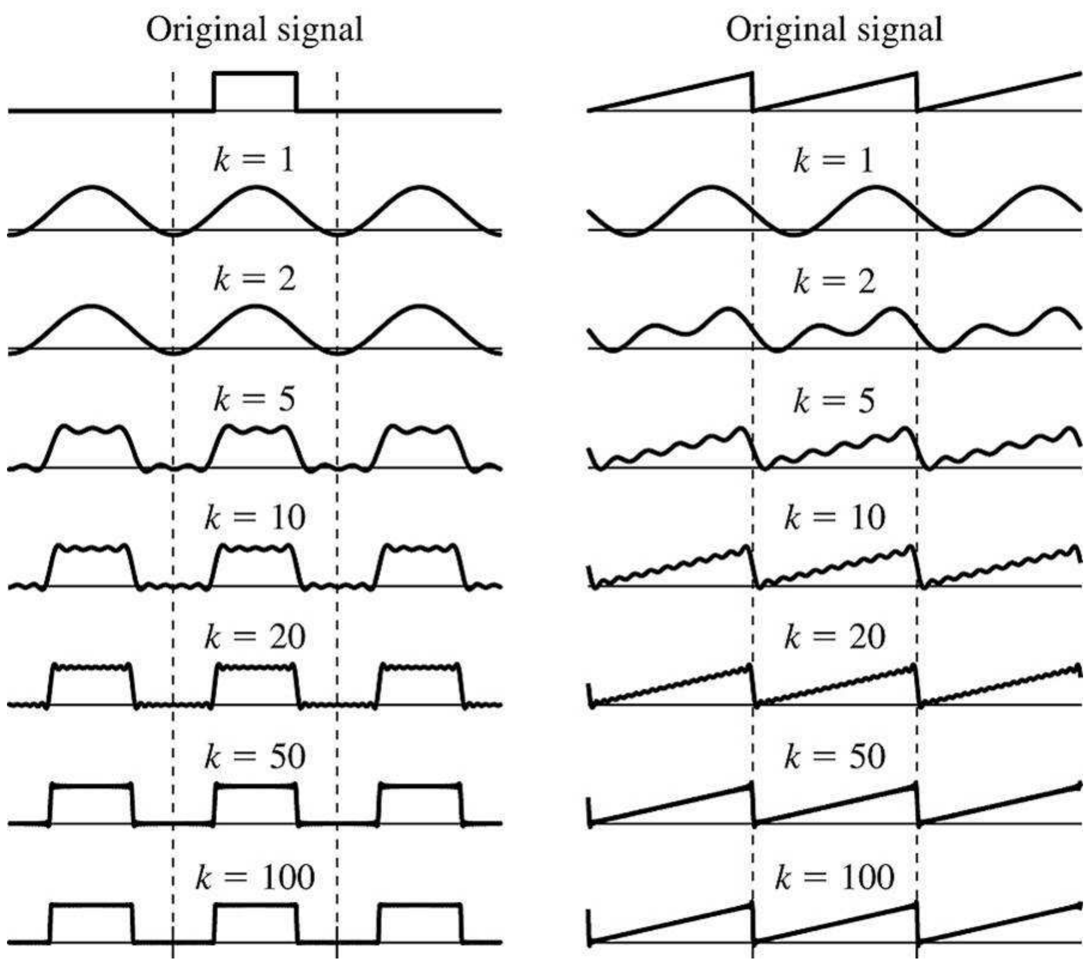


- $x_p(t)$ can be represented by its Fourier series:

$$x_p(t) = \sum_{k=-\infty}^{\infty}$$

- Obviously the original signal $x(t)$, has the same FS representation, **but only during** $t_0 \leq t \leq t_0 + T_f$.

FS representation



Example 1

Find the CTFS representation of $\sin(2\pi f_0 t)$. What is the harmonic function, $X[k]$?

Example 2

What is the harmonic function ($X[k]$) of $x(t) = \cos(8\pi t) + \cos(12\pi t)$?

- First we need to find the fundamental period or frequency of $x(t)$:

- Now we can represent $x(t) = \sum_k X[k] e^{j2\pi(kf_0)t}$

Example 3

Consider a periodic rectangular function with $T = 1$(s), defined over one period as

$$x(t) = \begin{cases} 1 & |t| < 1/4 \\ 0 & 1/4 < |t| < 1/2 \end{cases}$$

Find the CTFS coefficients (harmonic function), $X[k]$, of this signal.