ENSC380
Lecture 12

Objectives:

- Learn the important property of $X[k]$ when $x(t)$ is real:

$$X[k] = X^*[-k]$$

- Learn how to find the **trigonometric** (sinusoidal) CTFS for a real and periodic signal
- Learn the properties of CTFS

CTFS for real functions

- Recall: The Fourier series representation for $x(t)$, with period T_0 and frequency f_0 is:

$$x(t) =$$

- Conjugate both sides to write the FS for $x^*(t)$:

- If $x(t)$ is real, then $x(t) = x^*(t)$, this means:

$$X[k] =$$

Trigonometric FS

- For a real signal, $x(t)$, we can write its FS as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j2\pi(kf_0)t} = X[0] + \sum_{k=1}^{\infty} X[k]e^{j2\pi(kf_0)t} + \sum_{k=-\infty}^{-1} X[k]e^{j2\pi(kf_0)t}$$

$$= X[0] + \sum_{k=1}^{\infty} X[k]e^{j2\pi(kf_0)t} + \sum_{k=1}^{\infty}$$

- Finally writing $e^{j2\pi(kf_0)t} = \cos(2\pi(kf_0)t) + j\sin(2\pi(kf_0)t)$, we get:

$$x(t) = X[0] + \sum_{k=1}^{\infty} [X_c[k]\cos(2\pi(kf_0)t) + X_s[k]\sin(2\pi(kf_0)t)]$$

where:

$$X_c[k] = 2Re\{X[k]\} = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(2\pi(kf_0)t)dt$$

$$X_s[k] = -2Im\{X[k]\} = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(2\pi(kf_0)t)dt$$

Example 1

- Find the CTFS in the form of complex exponentials for $x(t) = \cos(2\pi f_0 t)$, using $T_0 = 1/f_0$ as the fundamental period of the Fourier series

- What is the trigonometric FS of $x(t)$? Does the result agree with the formulas for finding $X_c[k]$ and $X_s[k]$ from $X[k]$?

Example 2

In this example we see that the CTFS for a function can also be found using a fundamental period for the Fourir series, T_f, which is different from the fundamental period of $x(t)$.
Find the CTFS of $x(t) = \cos(2\pi f_0 t)$, using $T_f = 3T_0$ as the fundamental period of the Fourier series.

CTFS for Even/Odd signals

The integral for finding $X_c(t)$ and $X_s(t)$ can be calculated over any time interval of length T_0. Here consider $[-T_0/2, T_0/2]$:

$$X_c[k] = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cos(2\pi(kf_0)t)dt$$

$$X_s[k] = \frac{2}{T_0} \int_{T_0/2}^{T_0/2} x(t) \sin(2\pi(kf_0)t)dt$$

- If $x(t)$ is **even**:

- If $x(t)$ is **odd**:

CTFS Properties

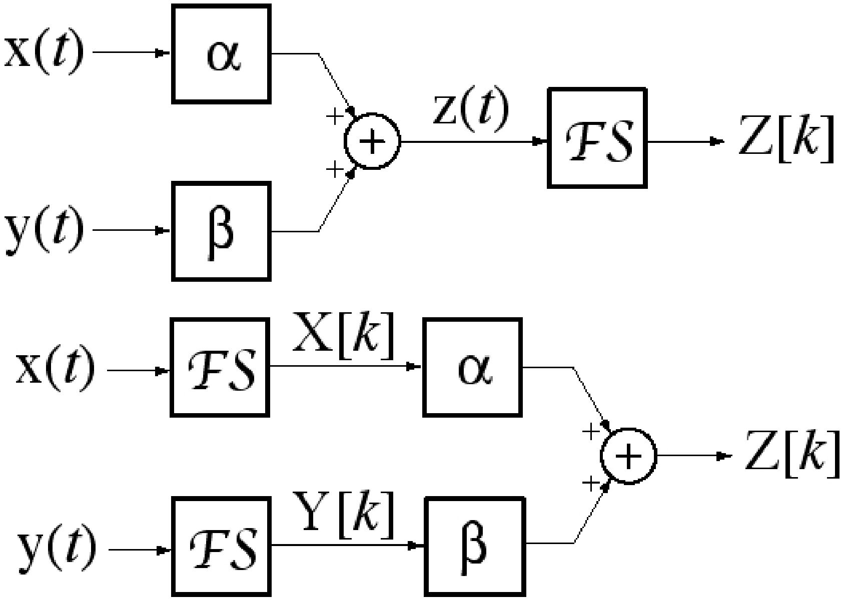
- In the next few slides we list the properties of CTFS. These slides are extracted from the Text's accompanying power point slides.
- Following that, we try to solve some CTFS problems using the FS pairs given in Appendix E of Text, and the properties of CTFS.
- Several more examples are given in the Text. See Section 4.5

Linearity

x(t) and y(t) both periodic with period T_0

Linearity

$$\alpha\, x(t) + \beta\, y(t) \xleftrightarrow{\text{FS}} \alpha\, X[k] + \beta\, Y[k]$$



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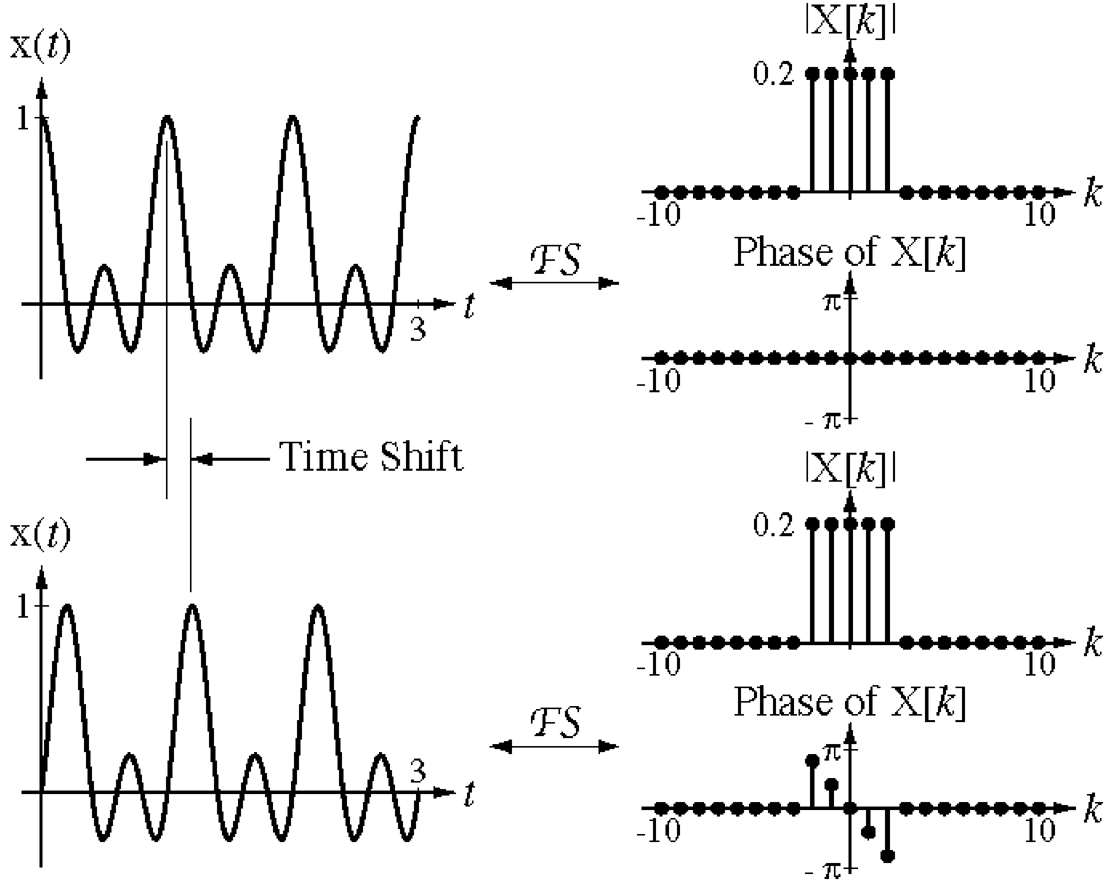
Time Shifting

Time Shifting

$$x(t - t_0) \overset{FS}{\longleftrightarrow} e^{-j2\pi(kf_0)t_0}\, X[k]$$

$$x(t - t_0) \overset{FS}{\longleftrightarrow} e^{-j(k\omega_0)t_0}\, X[k]$$



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Freq Shifting

Frequency Shifting (Harmonic Number Shifting)

$$e^{j2\pi(k_0 f_0)t}\,\mathrm{x}(t) \xleftrightarrow{\mathrm{FS}} \mathrm{X}[k-k_0]$$

$$e^{j(k_0 \omega_0)t}\,\mathrm{x}(t) \xleftrightarrow{\mathrm{FS}} \mathrm{X}[k-k_0]$$

A shift in frequency (harmonic number) corresponds to multiplication of the time function by a complex exponential.

Time Reversal $\quad \mathrm{x}(-t) \xleftrightarrow{\mathrm{FS}} \mathrm{X}[-k]$

Time Scaling

Let $z(t) = x(at)$, $a > 0$

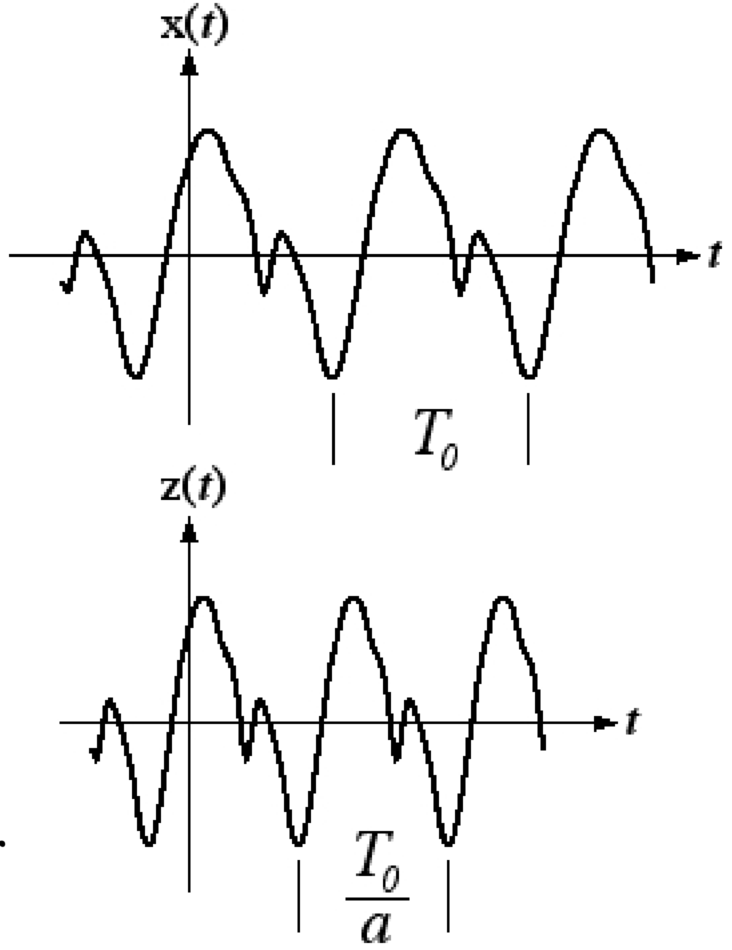
Case 1. $T_F = \dfrac{T_0}{a}$ for $z(t)$

$$Z[k] = X[k]$$

Case 2. $T_F = T_0$ for $z(t)$

If a is an integer,

$$Z[k] = \begin{cases} X\left[\dfrac{k}{a}\right], & \dfrac{k}{a} \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$



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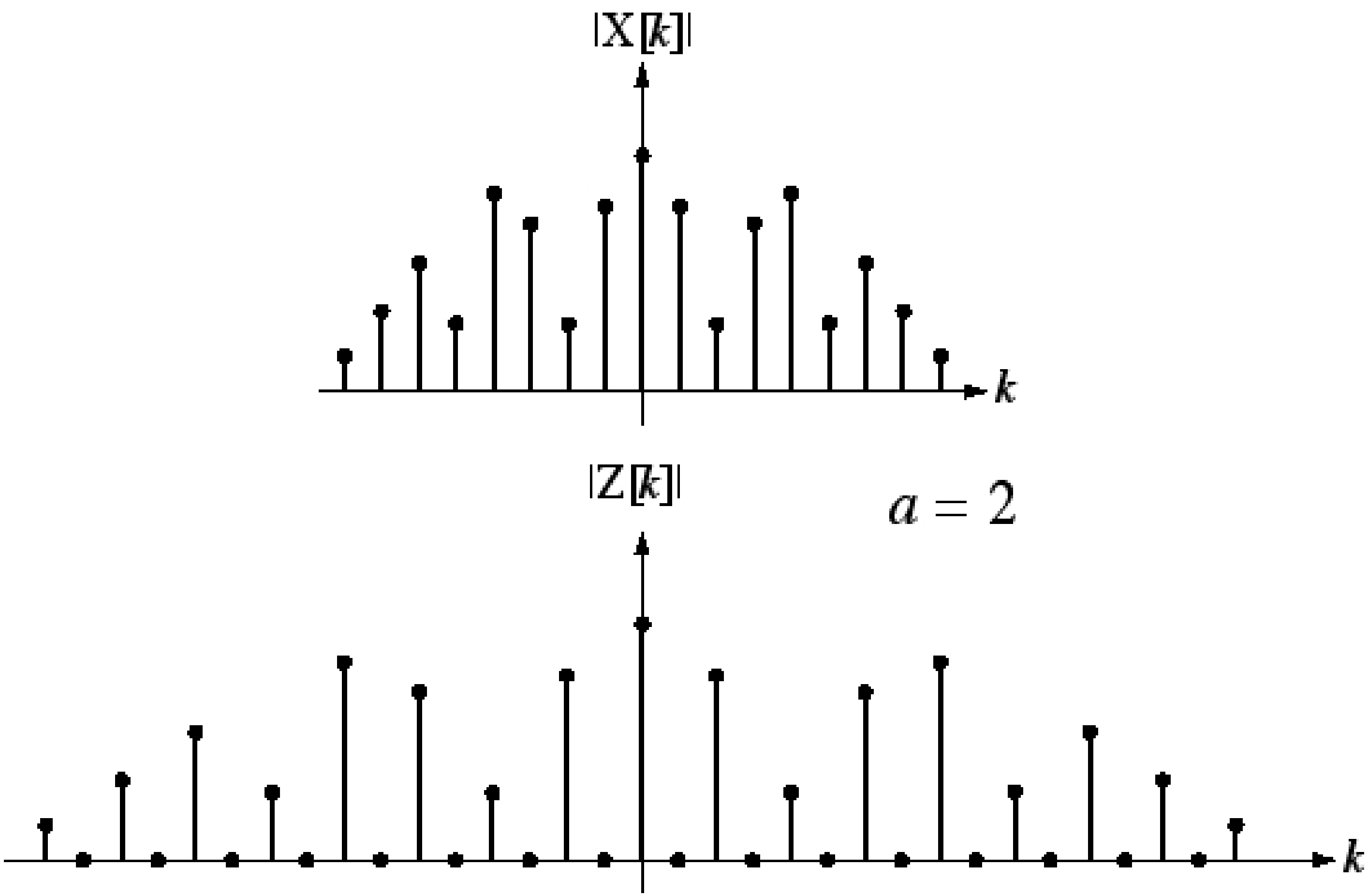
Time Scaling (Cont.)



Change of Representation Period

$$\text{With } T_F = T_{x0},\ \mathrm{x}(t) \xleftrightarrow{\text{FS}} \mathrm{X}[k]$$

$$\text{With } T_F = mT_{x0},\ \mathrm{x}(t) \xleftrightarrow{\text{FS}} \mathrm{X}_m[k]$$

$$\mathrm{X}_m[k] = \begin{cases} \mathrm{X}\left[\dfrac{k}{m}\right] & , \ \dfrac{k}{m} \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

(m is any positive integer)

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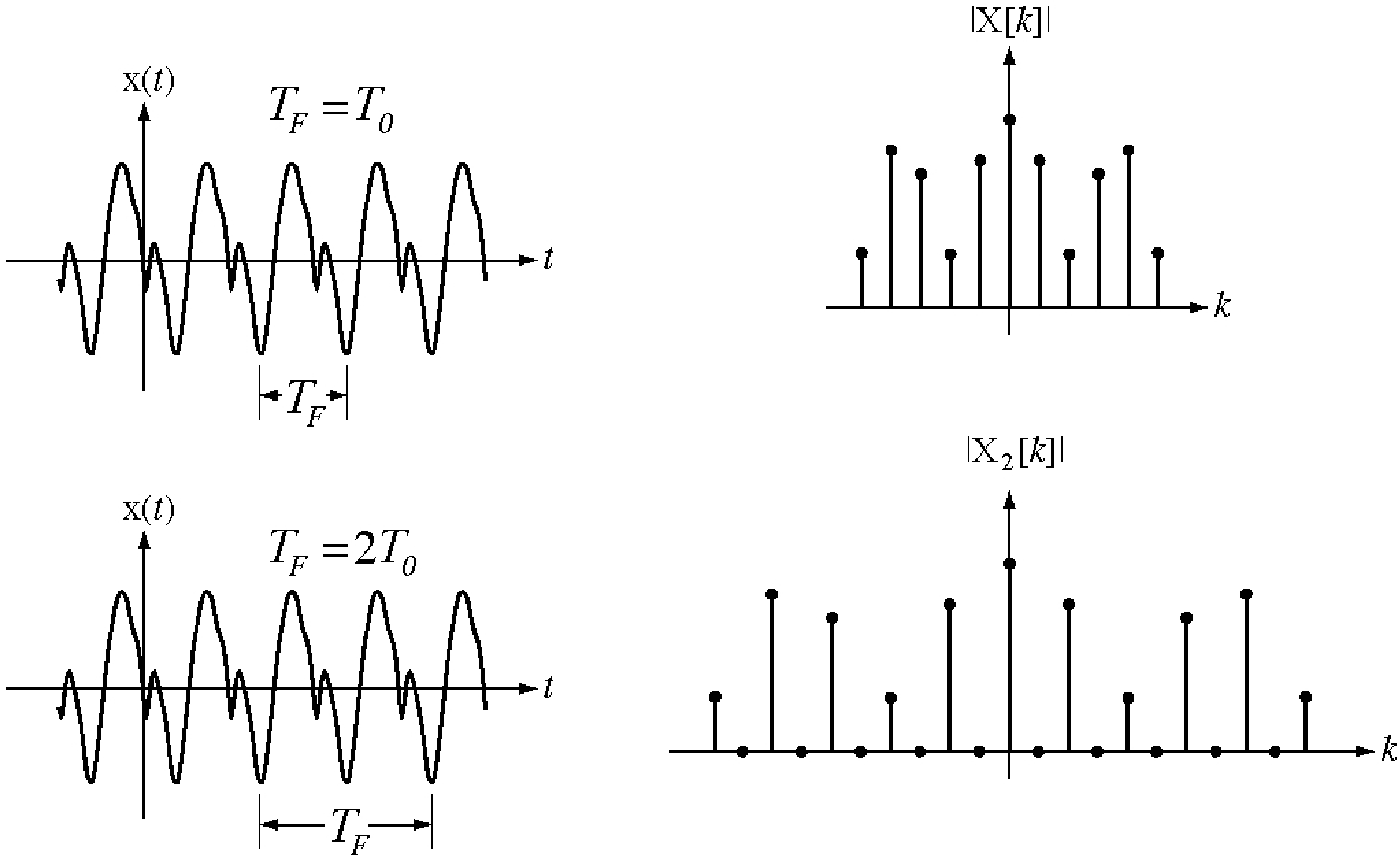
Change of Representation Period



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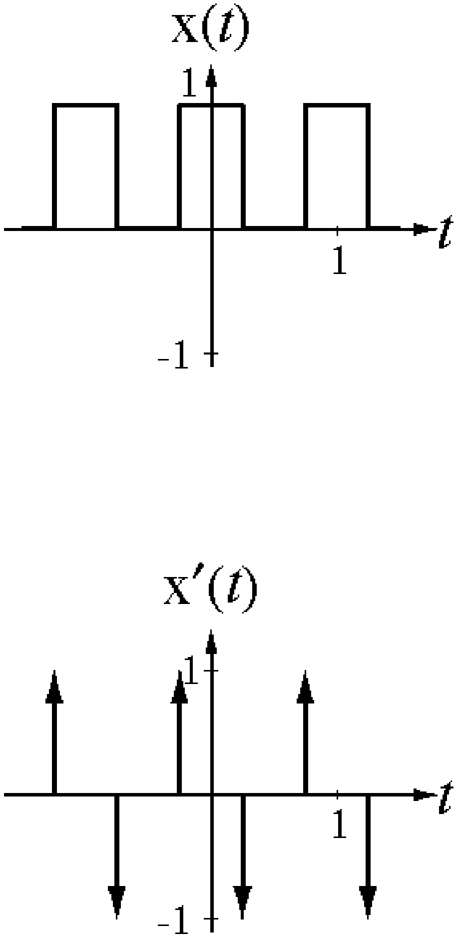
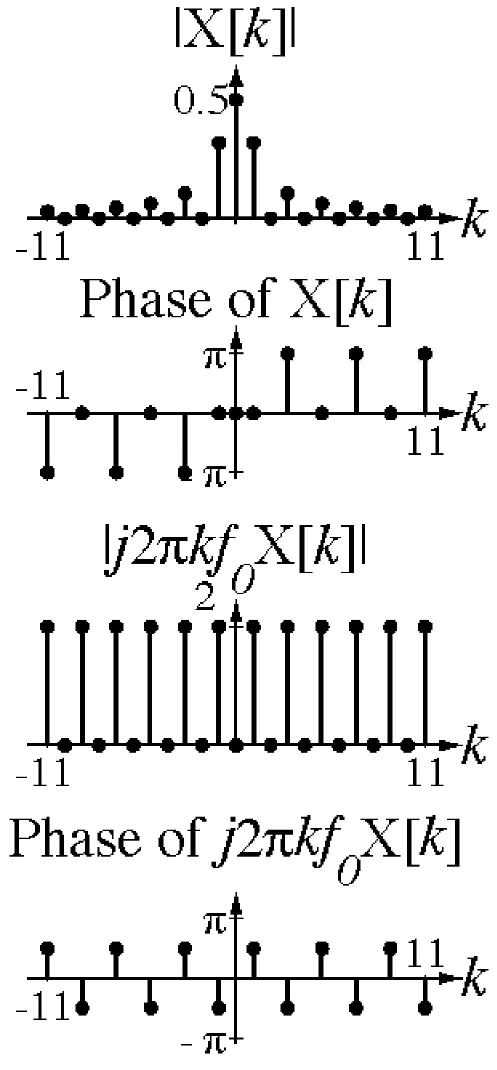
Time Differentiation

$$\frac{d}{dt}\left(x(t)\right) \xleftrightarrow{FS} j2\pi\left(kf_0\right)X[k]$$

$$\frac{d}{dt}\left(x(t)\right) \xleftrightarrow{FS} j\left(k\omega_0\right)X[k]$$





Time Integration

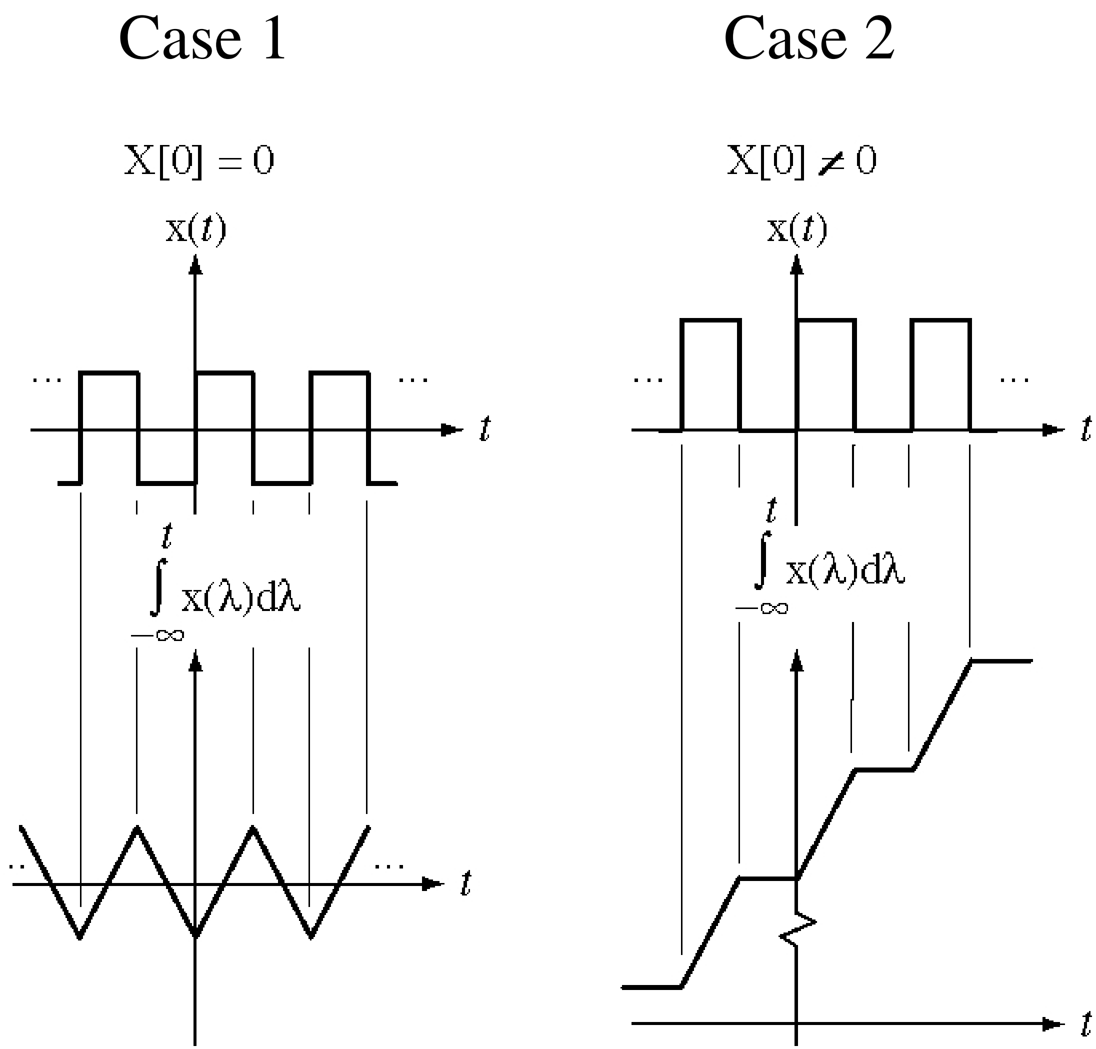
Case 1. $X[0] = 0$

$$\int_{-\infty}^{t} x(\lambda) d\lambda \overset{FS}{\longleftrightarrow} \frac{X[k]}{j2\pi(kf_0)}$$

$$\int_{-\infty}^{t} x(\lambda) d\lambda \overset{FS}{\longleftrightarrow} \frac{X[k]}{j(k\omega_0)}$$

Case 2. $X[0] \neq 0$

$$\int_{-\infty}^{t} x(\lambda) d\lambda \text{ is not periodic}$$



Multiplication-Convolution Duality

$$x(t)y(t) \xleftrightarrow{\text{FS}} X[k] * Y[k]$$

(The harmonic functions, X[k] and Y[k], must be based on the same representation period, T_F .)

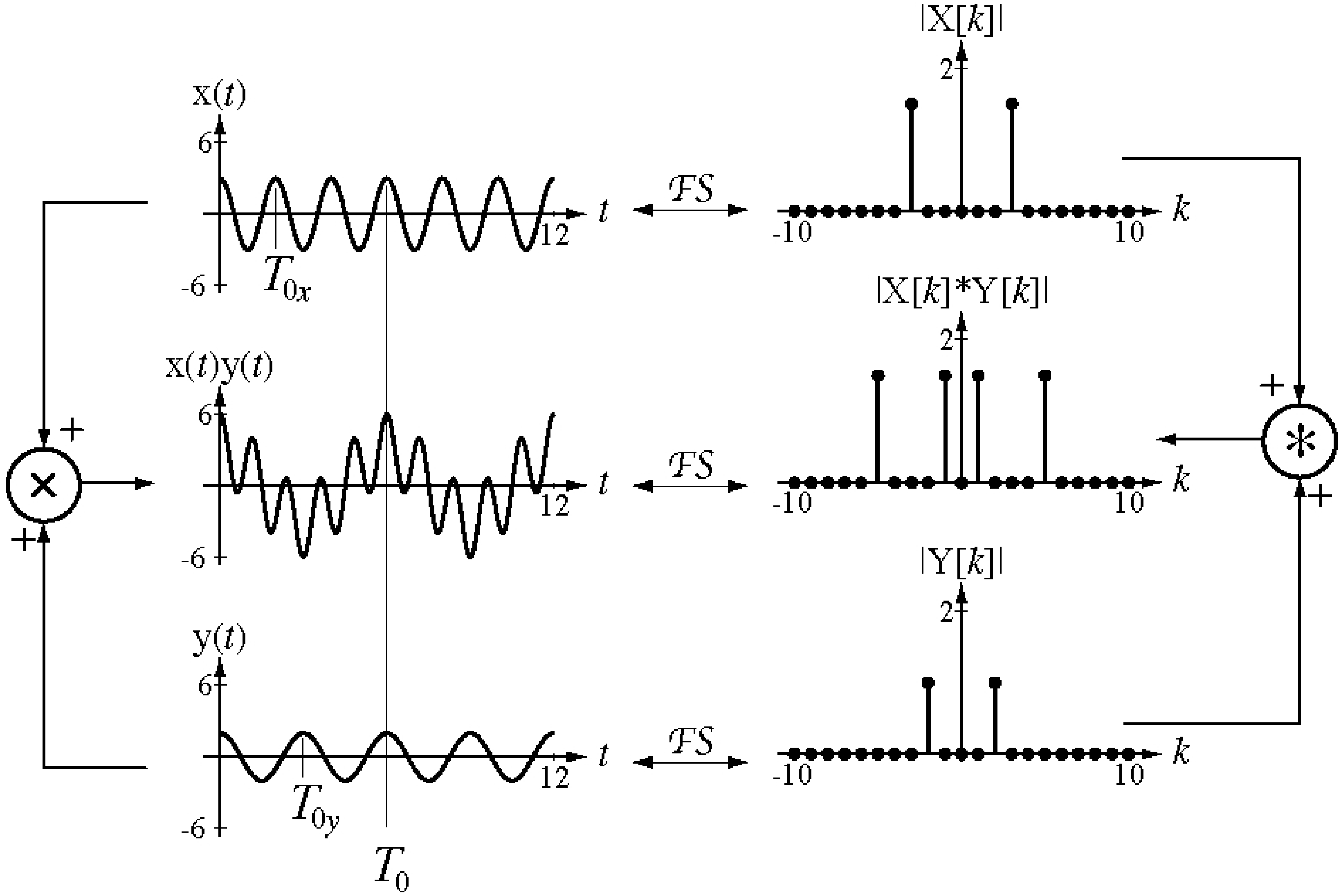
$$x(t) \circledast y(t) \xleftrightarrow{\text{FS}} T_0\, X[k] Y[k]$$

The symbol, $\circledast$, indicates *periodic convolution*.
Periodic convolution is defined mathematically by

$$x(t) \circledast y(t) = \int_{T_0} x(\tau) y(t-\tau) d\tau$$

$x(t) \circledast y(t) = x_{ap}(t) * y(t)$ where $x_{ap}(t)$ is any single period of $x(t)$

Multiplication-Convolution Duality



CTFS Properties

Conjugation

$$x^*(t) \xleftrightarrow{\text{FS}} X^*[-k]$$

Parseval's Theorem

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$$

The average power of a periodic signal is the sum of the average powers in its harmonic components.

Example 3

Using CTFS properties and Appendix E, find the CTFS of the following function, using the given T_f as the fundamental period of the Fourier series:

$$x(t) = 20\cos(100\pi(t - 0.005)) \qquad T_f = 1/50$$

Example 4

Using CTFS properties, find the CTFS of the following function, using the given T_f as the fundamental period of the Fourier series:

$$x(t) = rect(t) * comb(\frac{t}{4}) \qquad T_f = 4$$

Example 5

Using CTFS properties, find the CTFS of the following function, using the given T_f as the fundamental period of the Fourier series:

$$x(t) = \frac{d}{dt}(e^{-j10\pi t}) \qquad T_f = 1/5$$