ENSC380
Lecture 13

Objectives:

- Learn about the discrete-time Fourier Series (DTFS)
- Learn the properties of DTFS
- Solve examples

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DTFS

- DTFS is very similar in concept to CTFS, with minor differences.
- Consider a periodic DT signal with period N_0, i.e.,

$$x[n + kN_0] = x[n] \quad \forall k \quad \text{intger}$$

- The fundamental frequency of $x[n]$ is $F_0 = 1/N_0$
- DTFS for $x[n]$ is :

$$x[n] = \sum_{k=k_0}^{k_0+N_0-1} X[k]e^{j2\pi(kF_0)n} = \sum_{<N_0>} X[k]e^{j2\pi(kF_0)n}$$

where $\sum_{<N_0>}$ means the summation over any N_0 consecutive values of k.

- Difference with CTFS?
- Reason: $e^{j2\pi(kF_0)n}$ repeats itself with period N_0, because n is integer

DTFS coefficients

- The coefficients, $X[k]$ (harmonic function), are:

$$X[k] = \frac{1}{N_0} \sum_{n=n_0}^{n_0+N_0-1} x[n] e^{-j2\pi(kF_0)n} = \sum_{n=<N_0>} x[n] e^{-j2\pi(kF_0)n}$$

- Difference with CTFS? Summation instead of integral

Example 1

Find the DTFS representation of $x[n] = \text{comb}_{N_0}[n]$, using $N_f = N_0$ as the fundamental period of DTFS.

Example 2

Find the DTFS representation of $x[n] = \text{comb}_{N_0}[n]$, this time using $N_f = 3N_0$ as the fundamental period of DTFS.

Properties of DTFS

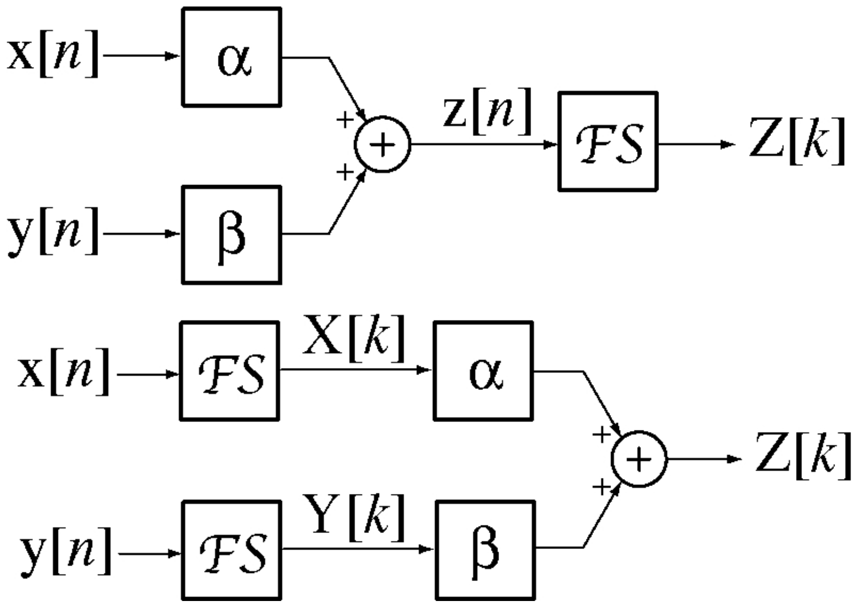
- Similar to the CTFS, here we list the properties of DTFS.
- We will then use these properties and Appendix E, to solve an example

DTFS Properties

Let a signal, x[n], have a fundamental period, N_{0x}, and let a signal, y[n], have a fundamental period, N_{0y}. Let the DTFS harmonic functions, each using the fundamental period as the representation time, N_F, be X[k] and Y[k]. In the properties to follow the two fundamental periods are the same unless otherwise stated.

Linearity

$$\alpha\,\mathrm{x}(t)+\beta\,\mathrm{y}(t) \xleftrightarrow{\mathcal{FS}} \alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]$$

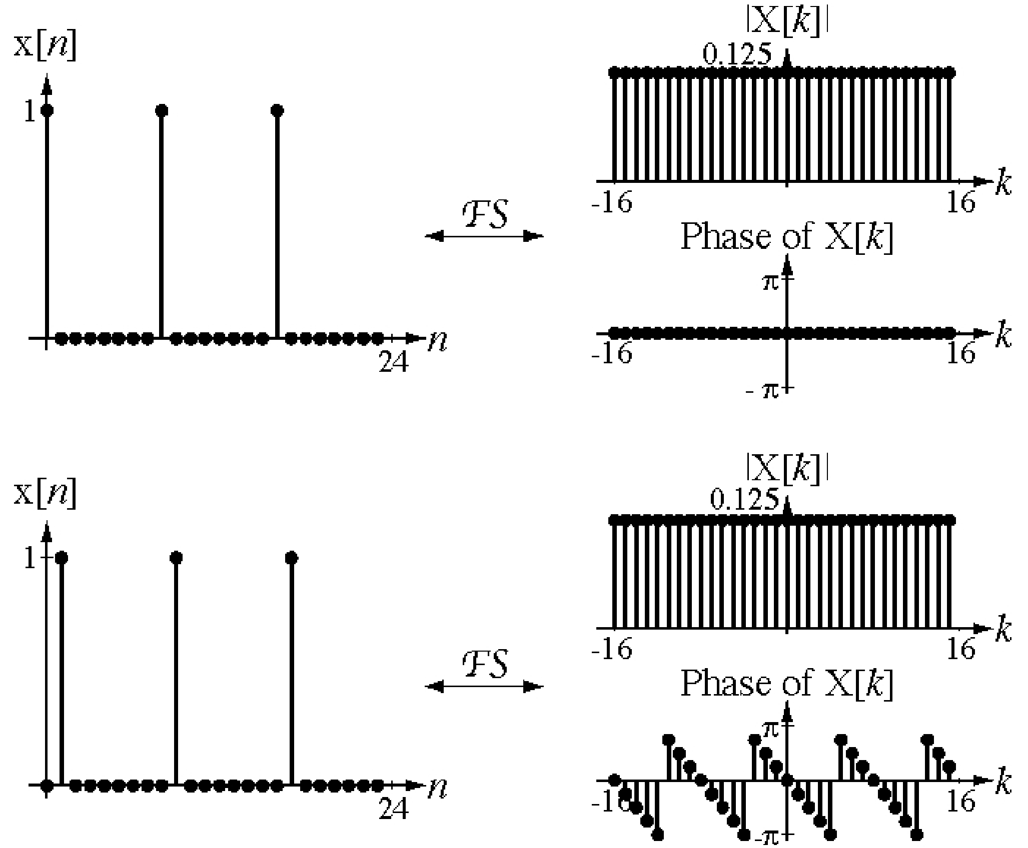


DTFS Properties

Time Shifting

$$x[n-n_0] \overset{FS}{\longleftrightarrow} e^{-j2\pi(kF_0)n_0}\, X[k]$$

$$x[n-n_0] \overset{FS}{\longleftrightarrow} e^{-j(k\Omega_0)n_0}\, X[k]$$



DTFS Properties

Frequency Shifting (Harmonic Number Shifting)

$$e^{j2\pi(k_0F_0)n}\,\mathrm{x}[n] \overset{\mathrm{FS}}{\longleftrightarrow} \mathrm{X}[k-k_0]$$

$$e^{j(k_0\Omega_0)n}\,\mathrm{x}[n] \overset{\mathrm{FS}}{\longleftrightarrow} \mathrm{X}[k-k_0]$$

Conjugation

$$\mathrm{x}^*[n] \overset{\mathrm{FS}}{\longleftrightarrow} \mathrm{X}^*[-k]$$

Time Reversal

$$\mathrm{x}[-n] \overset{\mathrm{FS}}{\longleftrightarrow} \mathrm{X}[-k]$$

DTFS Properties

Time Scaling

Let $\mathrm{z}[n] = \mathrm{x}[an],\ a > 0$

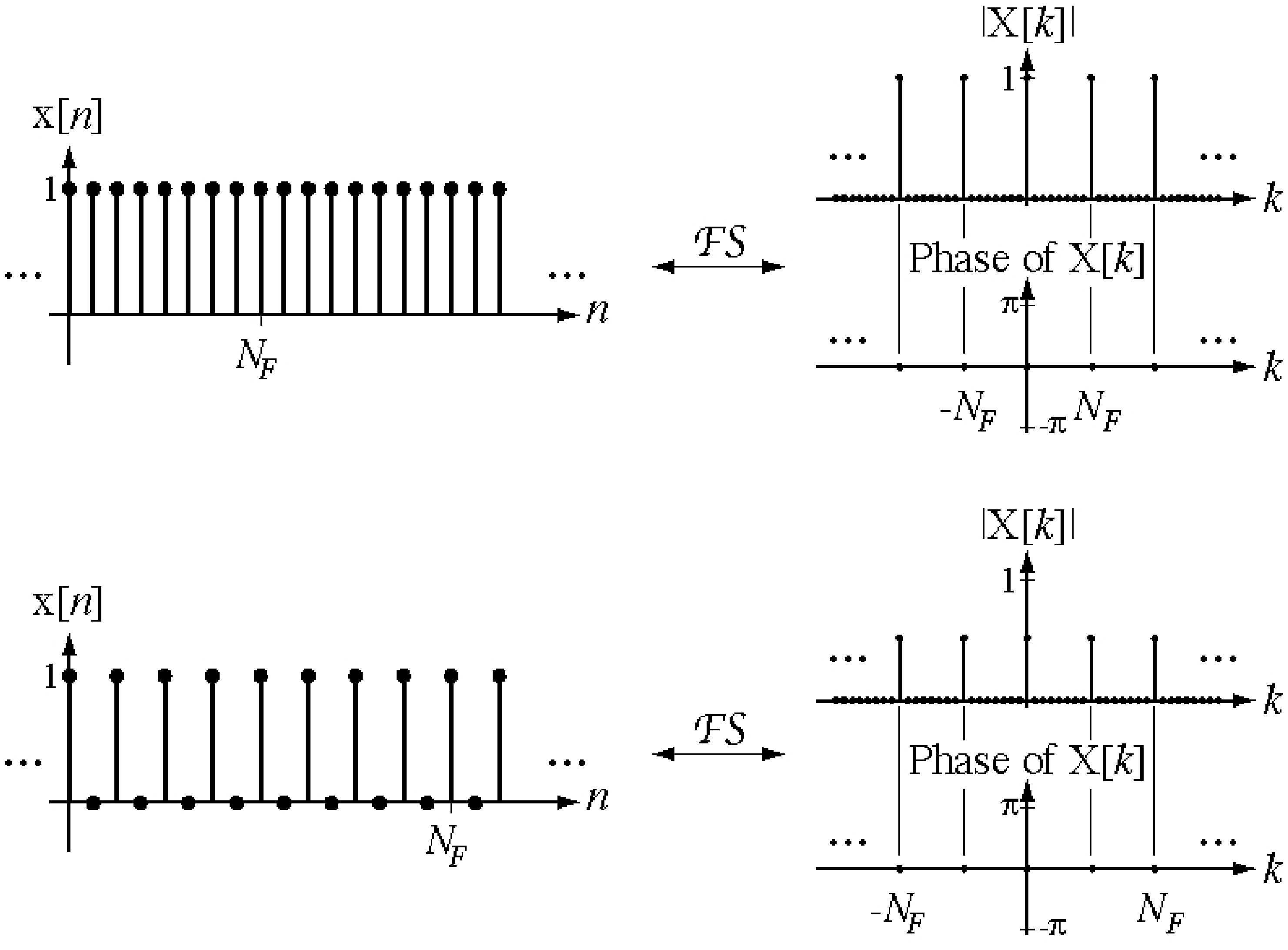
If a is not an integer, some values of z[n] are undefined and no DTFS can be found. If a is an integer (other than 1) then z[n] is a decimated version of x[n] with some values missing and there cannot be a unique relationship between their harmonic functions. However, if

$$\mathrm{z}[n] = \begin{cases} \mathrm{x}\left[\dfrac{n}{m}\right] & , \ \dfrac{n}{m} \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases} \qquad N_F = N_0$$

then

$$\mathrm{Z}[k] = \frac{1}{m}\mathrm{X}[k] \qquad N_F = mN_0$$

DTFS Properties



DTFS Properties

Change of Representation Period

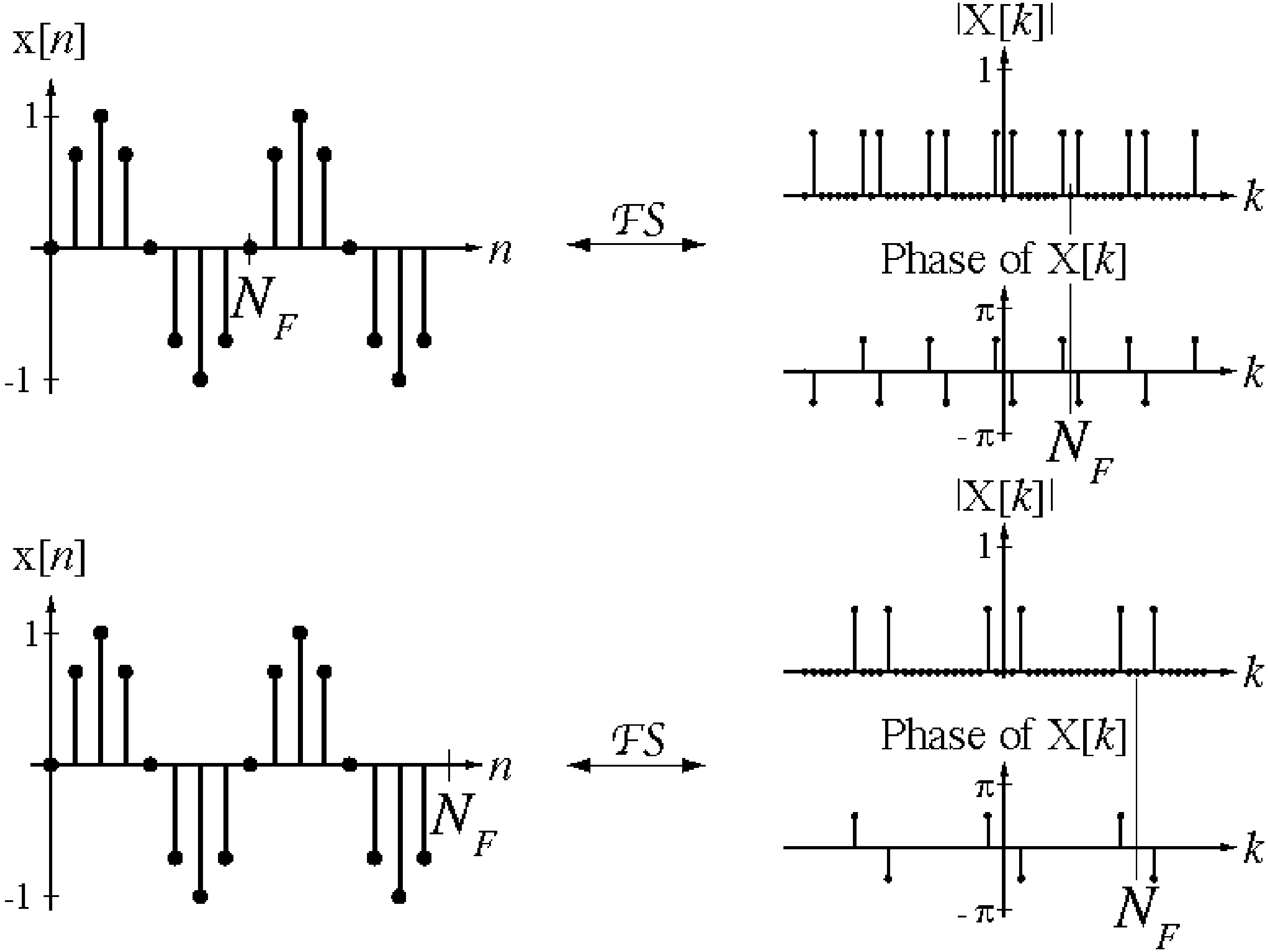
$$\text{With } N_F = N_{x0},\ \mathrm{x}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k]$$

$$\text{With } N_F = qN_{x0},\ \mathrm{x}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}_q[k]$$

$$\mathrm{X}_q[k] = \begin{cases} \mathrm{X}\left[\dfrac{k}{q}\right] &, \ \dfrac{k}{q} \text{ an integer} \\ 0 &, \ \text{otherwise} \end{cases}$$

(q is any positive integer)

DTFS Properties



DTFS Properties

Accumulation

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \xleftrightarrow{\mathcal{FS}} \frac{\mathrm{X}[k]}{1-e^{-j2\pi(kF_0)}} \quad , \quad k \neq 0$$

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \xleftrightarrow{\mathcal{FS}} \frac{\mathrm{X}[k]}{1-e^{-j(k\Omega_0)}} \quad , \quad k \neq 0$$

Parseval's Theorem

$$\frac{1}{N_0} \sum_{n=\langle N_0 \rangle} |\mathrm{x}[n]|^2 = \sum_{k=\langle N_0 \rangle} |\mathrm{X}[k]|^2$$

DTFS Properties

Multiplication-Convolution Duality

$$\mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{Y}[k] \circledast \mathrm{X}[k] = \sum_{q=\langle N_0 \rangle} \mathrm{Y}[q]\mathrm{X}[k-q]$$

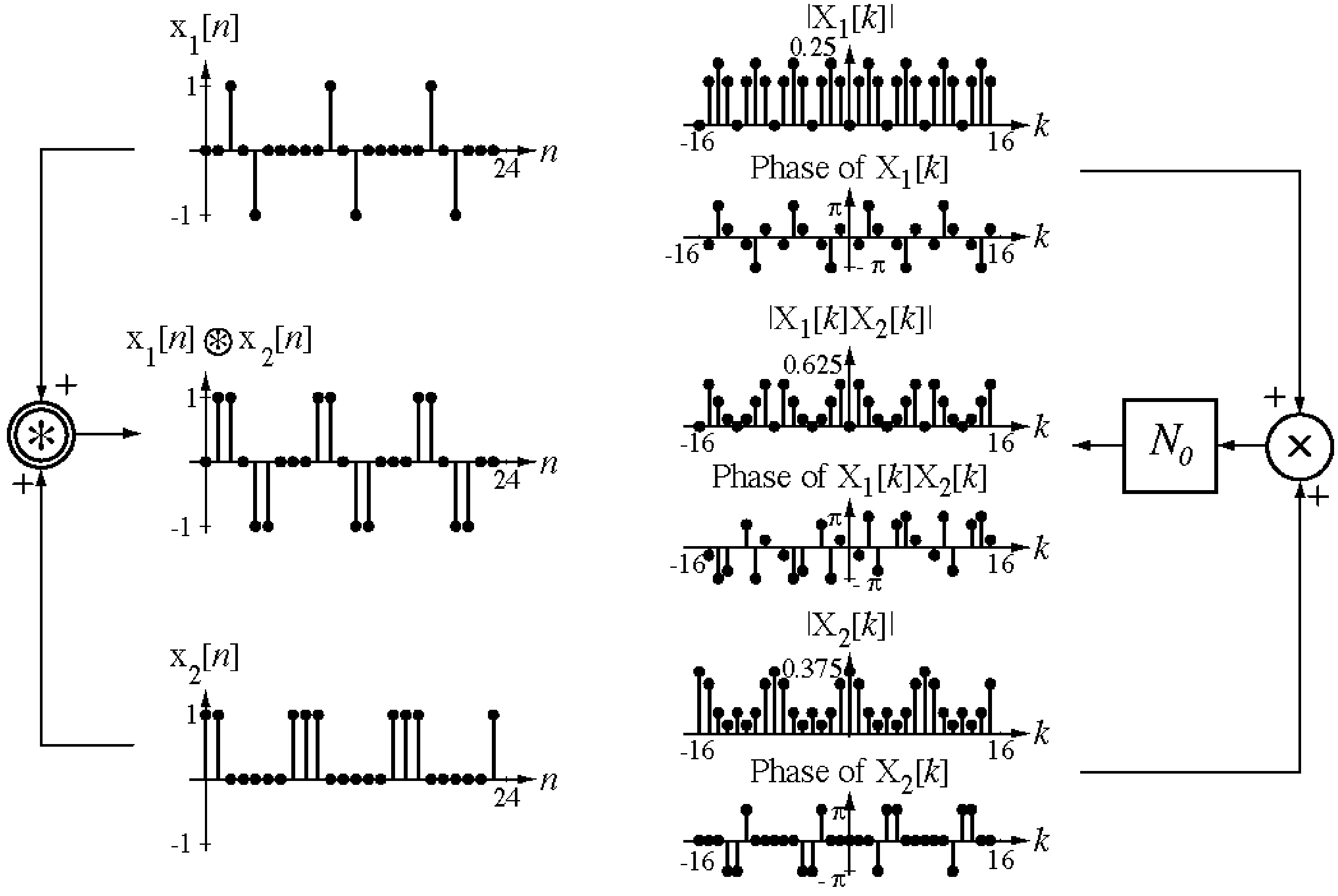
$$\mathrm{x}[n] \circledast \mathrm{y}[n] \xleftrightarrow{\mathcal{FS}} N_0\, \mathrm{Y}[k]\mathrm{X}[k]$$

First Backward Difference

$$\mathrm{x}[n] - \mathrm{x}[n-1] \xleftrightarrow{\mathcal{FS}} \left(1 - e^{-j2\pi(kF_0)}\right)\mathrm{X}[k]$$

$$\mathrm{x}[n] - \mathrm{x}[n-1] \xleftrightarrow{\mathcal{FS}} \left(1 - e^{-j(k\Omega_0)}\right)\mathrm{X}[k]$$

DTFS Properties



Example 3

Using Appendix E, and DTFS properties, find the DTFS representation of

$$x[n] = e^{-j(\pi n/8)} \otimes \text{comb}_{24}[n] \qquad N_f = 48$$