ENSC380
Lecture 15

Objectives:

- Learn about the continuous time Fourier Transform (CTFT) for
  - Aperiodic signals
  - Periodic signals

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Recap of CTFS

- We know that a periodic signal, $x_p(t)$, has a CTFS representation in complex exponential or trigonometric forms:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{2\pi(kf_0)t} = X[0]+\sum_{k=1}^{\infty} [X_c[k]\cos(2\pi(kf_0)t) + X_s[k]\sin(2\pi(kf_0)t)]$$

- Example: The CTFS representations of $10\cos(2\pi 100t)$ in exponential and trigonometric forms are:

  Which means, $\cos(2\pi 100t)$ has a frequency component of (Hz) with a weight of .

- Example: If the CTFS representation of a signal is given as:

$$x(t) = 4 + 2\cos(2\pi(300)t) + 0.5\sin(j2\pi(400)t)$$

  It means: The signal $x(t)$ has a frequency component of (Hz) with a weight of , and a frequency component of (Hz) with a weight of , and a frequency component of (Hz) with a weight of .

CT-Fourier Transform (CTFT)

- CTFS can only be found for periodic signals or over a finite time interval for aperiodic signals.
- Can we find out about the frequency contents of a complete aperiodic signal?
- Yes, this can be done using a representation called the continuous time Fourier transform (CTFT):

$$X(f) = \mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \mathcal{F}^{-1}(X(f)) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Notations:

$$X(f) = \mathcal{F}(x(t)) \quad \text{or} \quad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$

- $X(f)$ can also be written in terms of the radian frequency, by replacing f with $\omega/2\pi$ and is shown with $X(j\omega)$.

Example 1

Find the Fourier transform of $x(t) = \text{rect}(t)$

Example 2

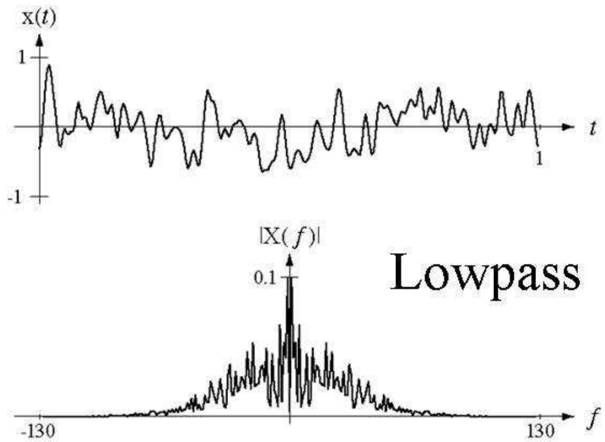
Find the Inverse FT of $X(f) = \frac{1}{j\pi f}$.

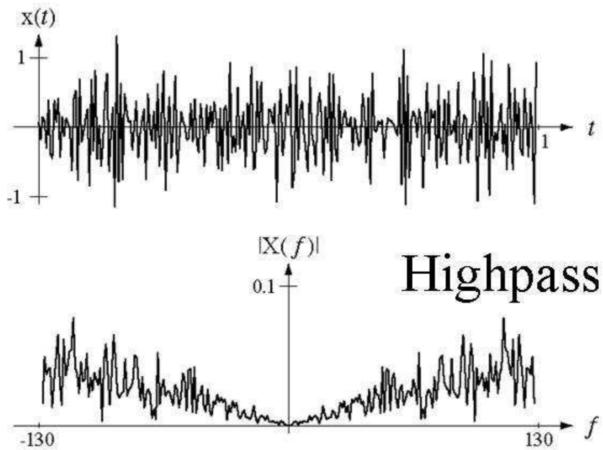
Interpretation of CTFT

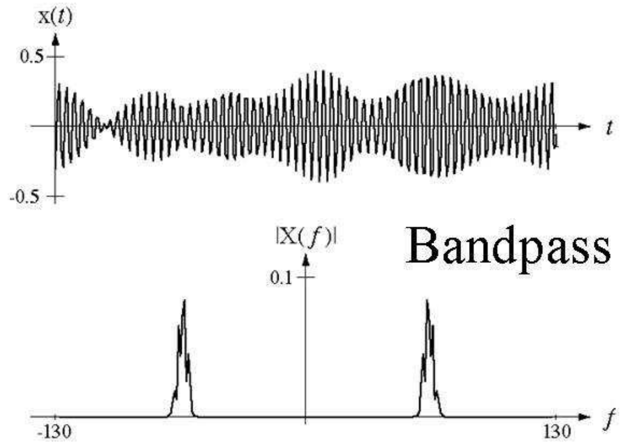
- The CTFS of a periodic signal with fundamental frequency f_0, indicates the weight of the frequency components of the signal at frequencies 0, f_0, $2f_0$, $\ldots$, kf_0, $\ldots$
- The CTFT of a signal (aperiodic) is a continuous function of the frequency f. $X(f)$ between two frequencies, f_1, and $f_1 + df$, is an indication of how much of the signal's energy is contained in this range of frequencies.
- Let's look at the FT of some signals using Text book's Matlab concept simulator.

Frequency Content

Signals can be categorized based on their frequency content, i.e., the range of frequencies their Fourier transform covers (see next slide)







Convergence of CTFT

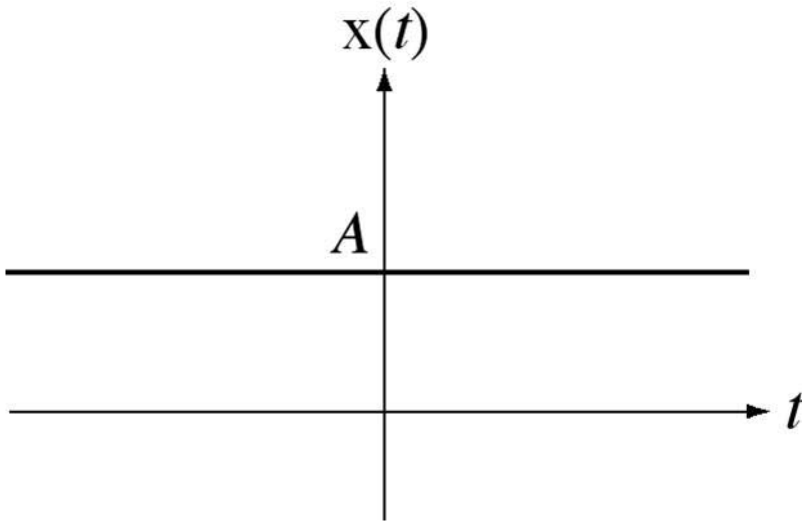
- The condition for convergence of CTFT, is similar to the condition for CTFS. i.e., the signal $x(t)$ should be **absolutely integrable**, but here over the interval of $[-\infty\infty]$:

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

- There are two more conditions for convergence of CTFS and CTFT, but these are always valid for practical signals.
- For some signals which are not absolutely integrable over $[-\infty\infty]$, such as periodic signals, or a constant value signal ($x(t) = A$), we can still find their CTFT if we allow $\delta(f)$ in the Fourier transform.

CTFT for $x(t) = A$

- For $x(t) = A$, we have $X(f) = \int_{-\infty}^{\infty} Ae^{-j2\pi ft}dt$. This integral does not converge.

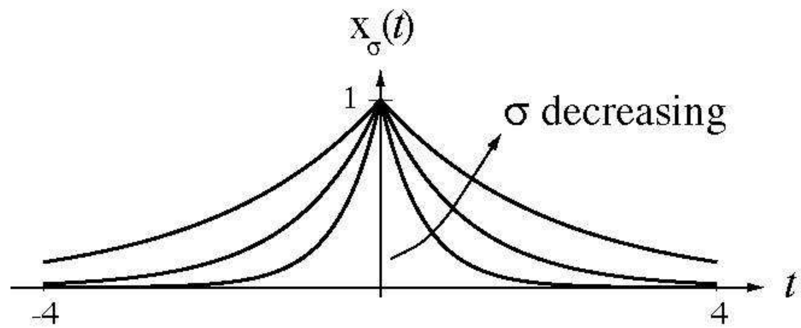


- But now consider $x_\sigma(t) = Ae^{-\sigma|t|} \quad \sigma > 0$. For this signal:

$$X_\sigma(f) = \int_{-\infty}^{\infty} Ae^{-\sigma|t|}e^{-j2\pi ft}dt = A\frac{2\sigma}{\sigma^2 + (2\pi f)^2}$$

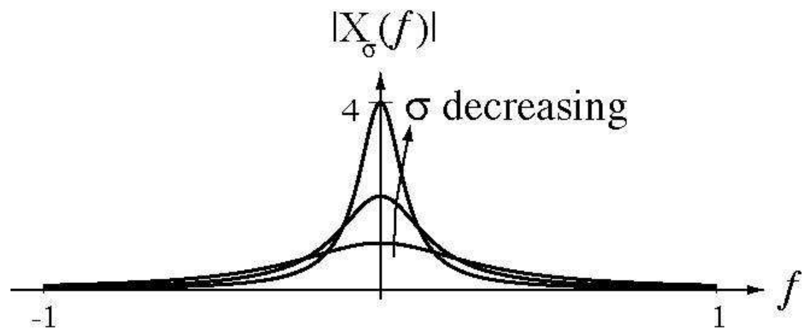
CTFT for $x(t) = A$ (Cont.)

- Now let's decrease σ towards 0:



- For $f \neq 0$; $\lim_{\sigma \to 0} X_\sigma(f) =$
- For $f = 0$; $\lim_{\sigma \to 0} X_\sigma(f) =$
- We can show that the area under $X_\sigma(f)$ is :

$$\int_{-\infty}^{\infty} X_\sigma(f) df = A \quad \text{(independent of } \sigma\text{)}$$



- We can conclude that: $\lim_{\sigma \to 0} X_\sigma(f) = \delta(f)$
- Important Fourier Pairs:

$$A \overset{\mathcal{F}}{\longleftrightarrow} A\delta(f) \ \text{ or } \ 1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(f)$$

- Using a similar approach we can show that:

$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f - f_0)$$

CTFT for $\cos(.)$ and $\sin(.)$

- Using the last result above and assuming we know that CTFT is **Linear** we can show that:

$$\cos(2\pi f_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

$$\sin(2\pi f_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

Graphical Illustration