ENSC380
Lecture 16

Objectives:

- Learn the properties of CTFT
- Find CTFT of different signals using these properties

CTFT Properties

- Here we list the properties of CTFT.

- Using these properties and the FT pairs listed in Appendix E, we should be able to find the CTFT of many signals.

CTFT Properties

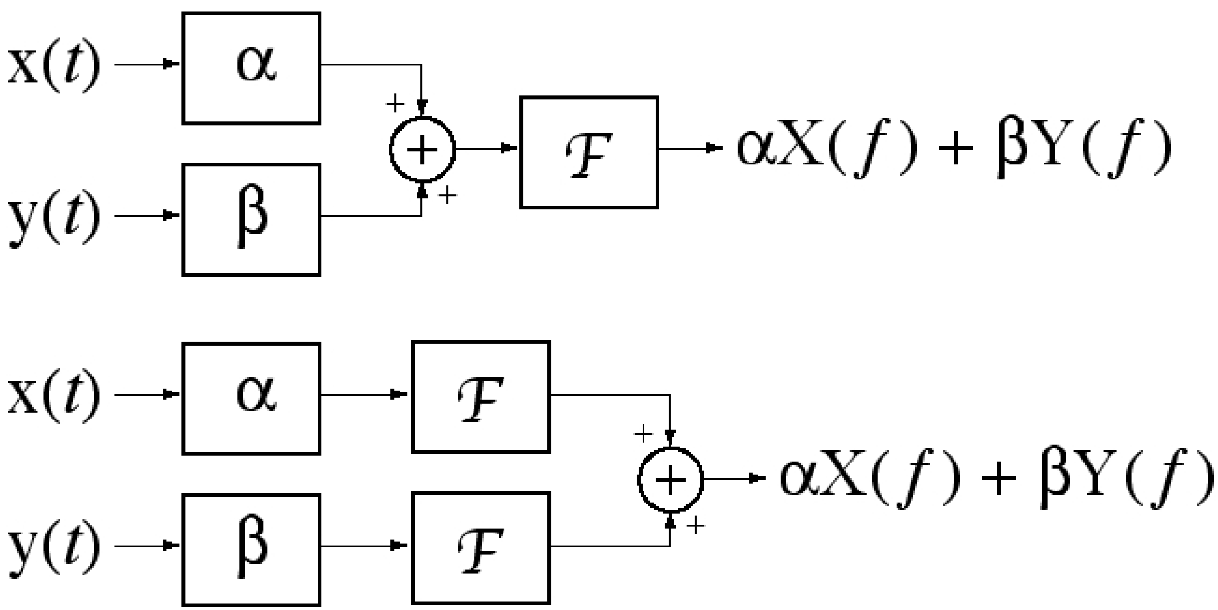
If $\mathrm{F}\left(\mathrm{x}(t)\right) = \mathrm{X}(f)$ or $\mathrm{X}(j\omega)$ and $\mathrm{F}\left(\mathrm{y}(t)\right) = \mathrm{Y}(f)$ or $\mathrm{Y}(j\omega)$

then the following properties can be proven.

Linearity

$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \overset{\mathrm{F}}{\longleftrightarrow} \alpha\,\mathrm{X}(f) + \beta\,\mathrm{Y}(f)$$

$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \overset{\mathrm{F}}{\longleftrightarrow} \alpha\,\mathrm{X}(j\omega) + \beta\,\mathrm{Y}(j\omega)$$



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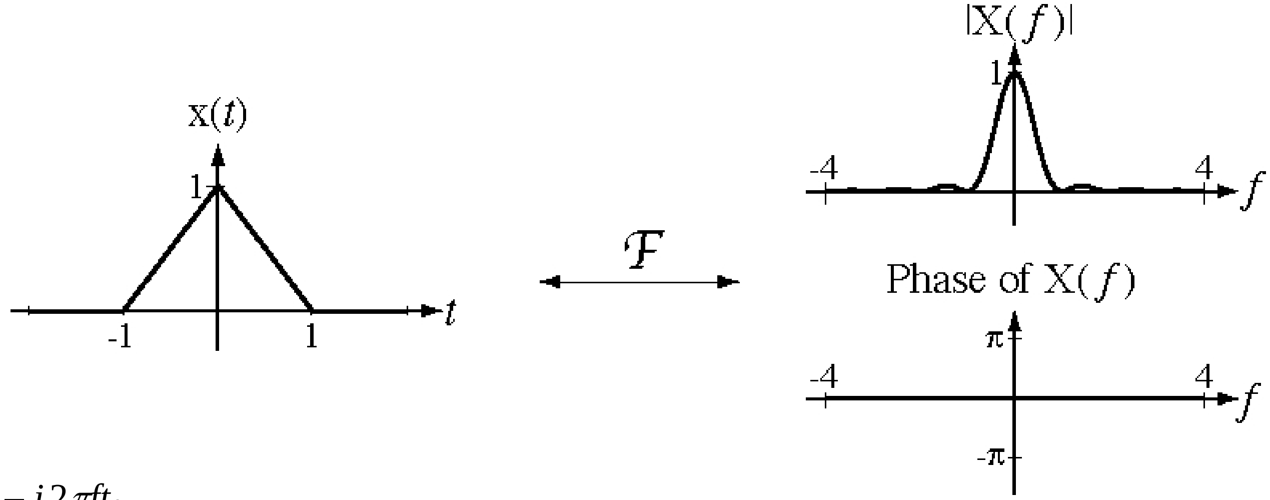
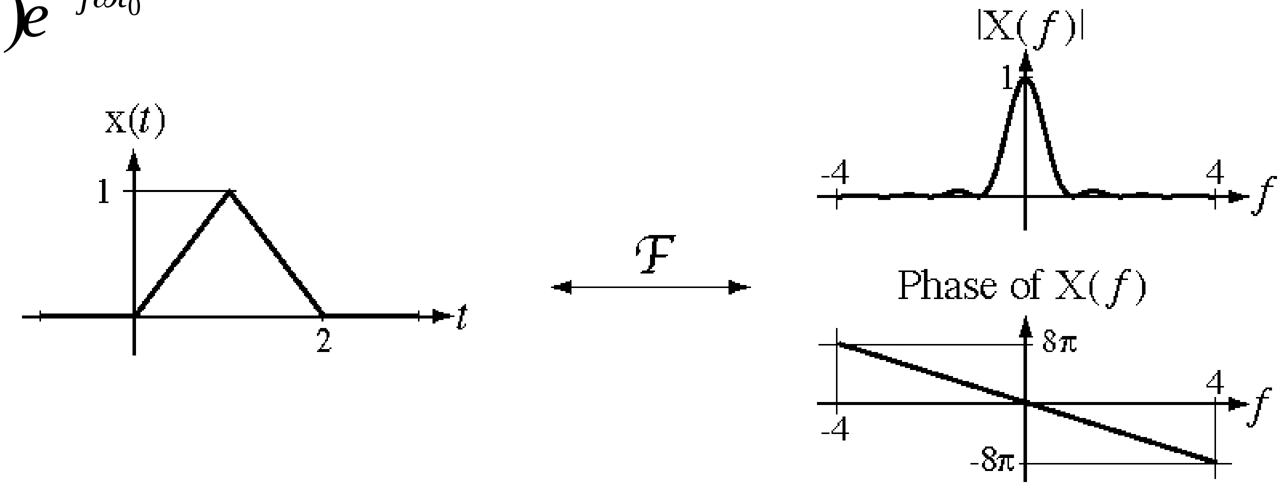
CTFT Properties



Time Shifting

$$x(t-t_0) \xleftrightarrow{F} X(f)e^{-j2\pi f t_0}$$

$$x(t-t_0) \xleftrightarrow{F} X(j\omega)e^{-j\omega t_0}$$

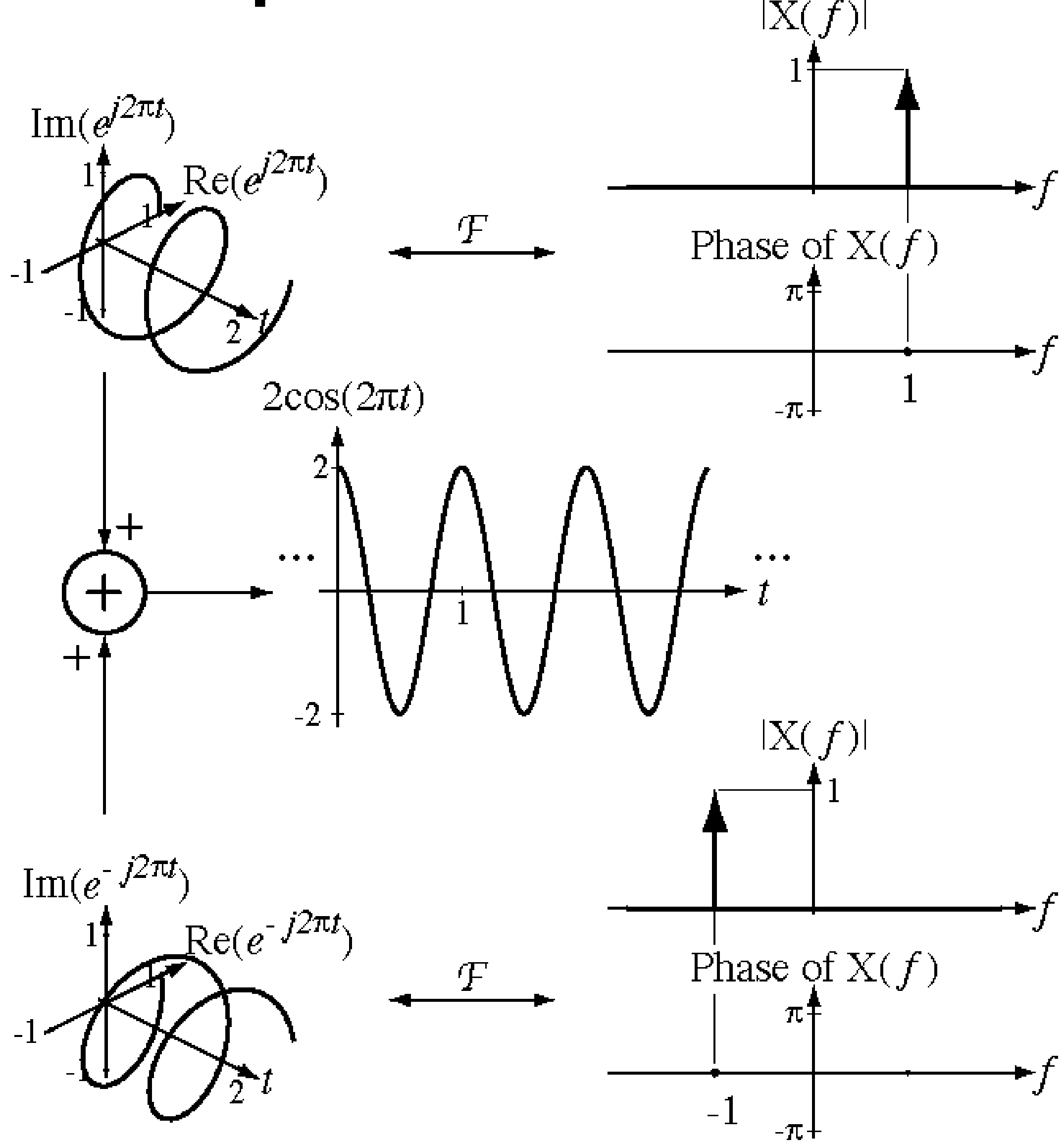


CTFT Properties

Frequency Shifting

$$x(t)e^{+j2\pi f_0 t} \overset{F}{\longleftrightarrow} X(f-f_0)$$

$$x(t)e^{+j\omega_0 t} \overset{F}{\longleftrightarrow} X(\omega-\omega_0)$$



CTFT Properties

Time Scaling

$$\mathrm{x}(at) \overset{\mathrm{F}}{\longleftrightarrow} \frac{1}{|a|}\mathrm{X}\left(\frac{f}{a}\right)$$

$$\mathrm{x}(at) \overset{\mathrm{F}}{\longleftrightarrow} \frac{1}{|a|}\mathrm{X}\left(j\frac{\omega}{a}\right)$$

Frequency Scaling

$$\frac{1}{|a|}\mathrm{x}\left(\frac{t}{a}\right) \overset{\mathrm{F}}{\longleftrightarrow} \mathrm{X}(af)$$

$$\frac{1}{|a|}\mathrm{x}\left(\frac{t}{a}\right) \overset{\mathrm{F}}{\longleftrightarrow} \mathrm{X}(ja\omega)$$

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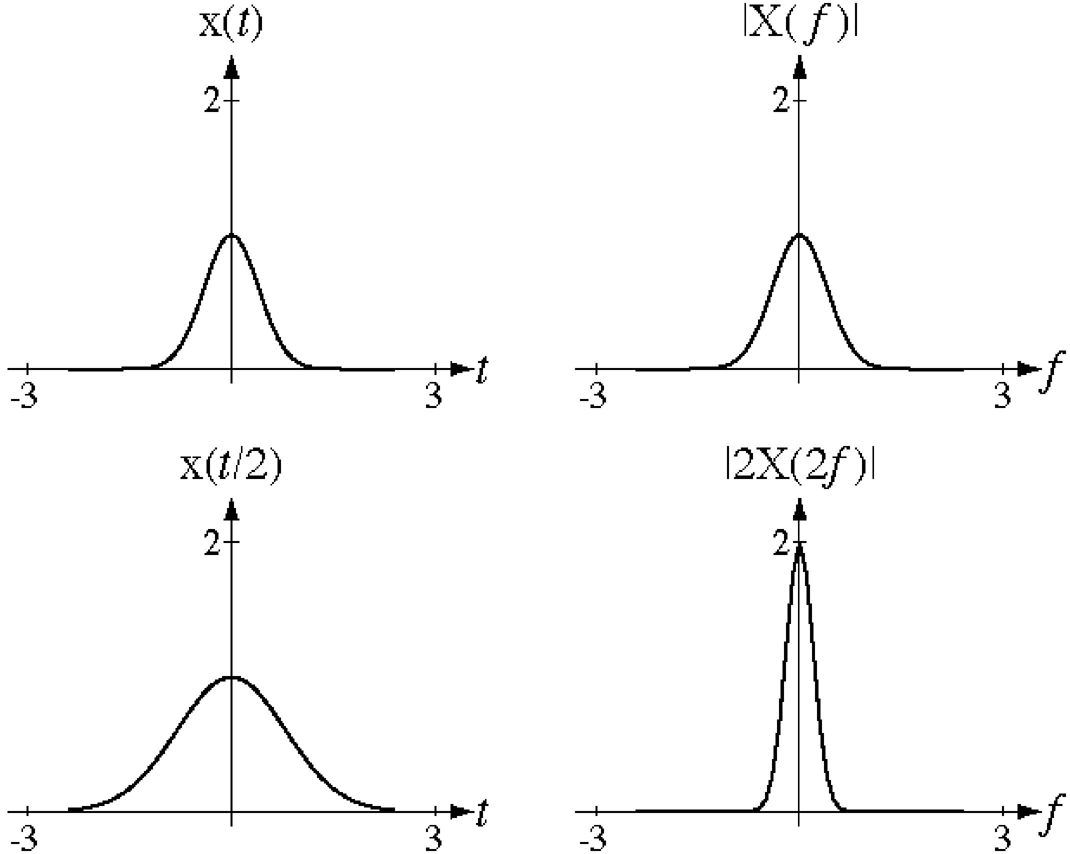
The "Uncertainty" Principle

The time and frequency scaling properties indicate that if a signal is expanded in one domain it is compressed in the other domain. This is called the "uncertainty principle" of Fourier analysis.

$$e^{-\pi t^2} \stackrel{F}{\longleftrightarrow} e^{-\pi f^2}$$

$$e^{-\pi\left(\frac{t}{2}\right)^2} \stackrel{F}{\longleftrightarrow} 2e^{-\pi(2f)^2}$$



CTFT Properties

Transform of a Conjugate

$$x^*(t) \overset{F}{\longleftrightarrow} X^*(-f)$$

$$x^*(t) \overset{F}{\longleftrightarrow} X^*(-j\omega)$$

Multiplication-Convolution Duality

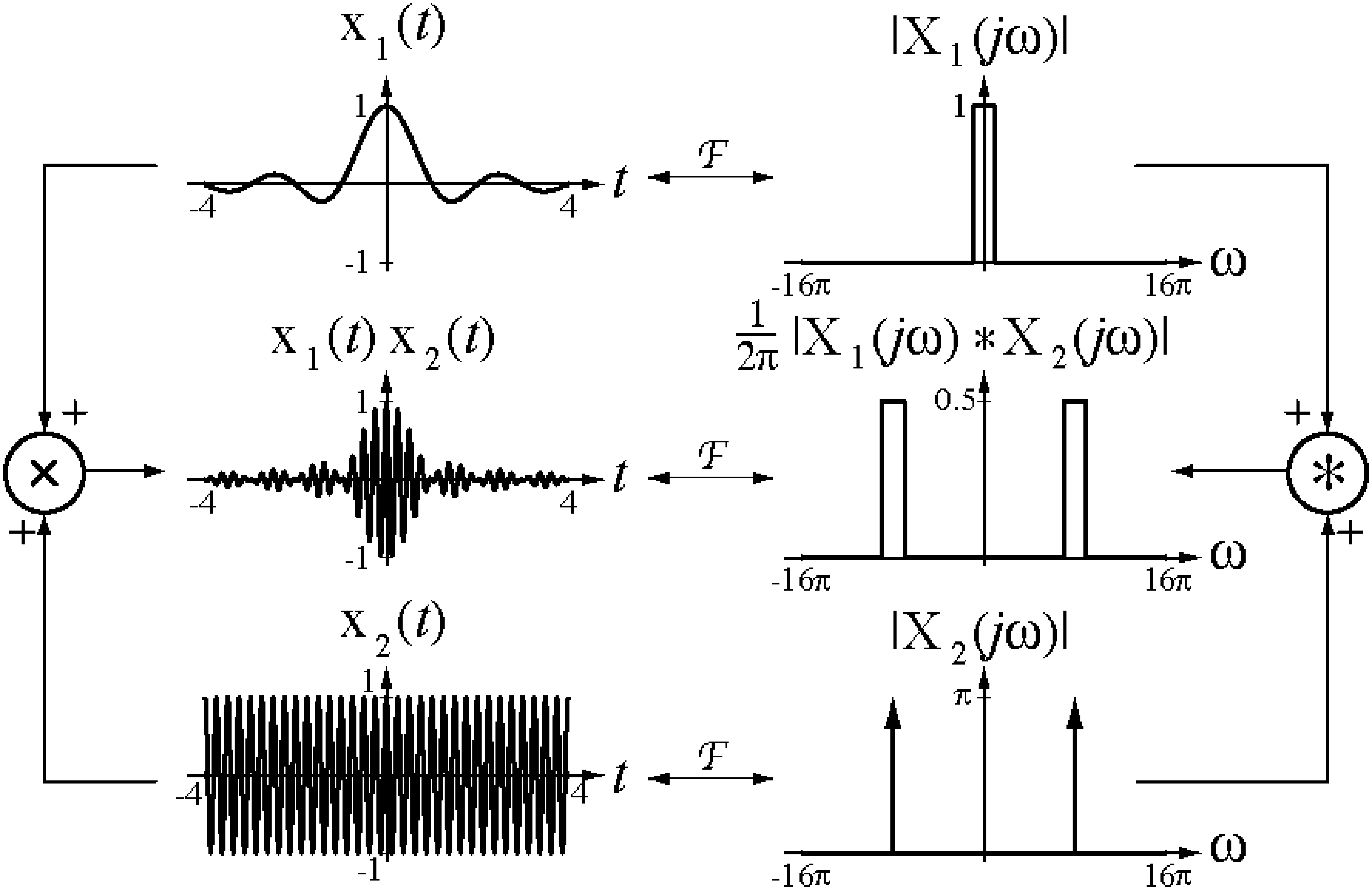
$$x(t) * y(t) \overset{F}{\longleftrightarrow} X(f)Y(f)$$

$$x(t) * y(t) \overset{F}{\longleftrightarrow} X(j\omega)Y(j\omega)$$

$$x(t)y(t) \overset{F}{\longleftrightarrow} X(f) * Y(f)$$

$$x(t)y(t) \overset{F}{\longleftrightarrow} \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

CTFT Properties



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CTFT Properties

An important consequence of multiplication-convolution duality is the concept of the *transfer function*.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = h(t) * x(t) \qquad X(f) \rightarrow \boxed{H(f)} \rightarrow Y(f) = H(f)X(f)$$

In the frequency domain, the cascade connection multiplies the transfer functions instead of convolving the impulse responses.

$$X(f) \rightarrow \boxed{H_1(f)} \rightarrow X(f)H_1(f) \rightarrow \boxed{H_2(f)} \rightarrow Y(f) = X(f)H_1(f)H_2(f)$$

$$X(f) \rightarrow \boxed{H_1(f)H_2(f)} \rightarrow Y(f)$$

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CTFT Properties

Time Differentiation

$$\frac{d}{dt}\left(x(t)\right) \overset{F}{\longleftrightarrow} j2\pi f\, X(f)$$

$$\frac{d}{dt}\left(x(t)\right) \overset{F}{\longleftrightarrow} j\omega\, X(j\omega)$$

Modulation

$$x(t)\cos(2\pi f_0 t) \overset{F}{\longleftrightarrow} \frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right]$$

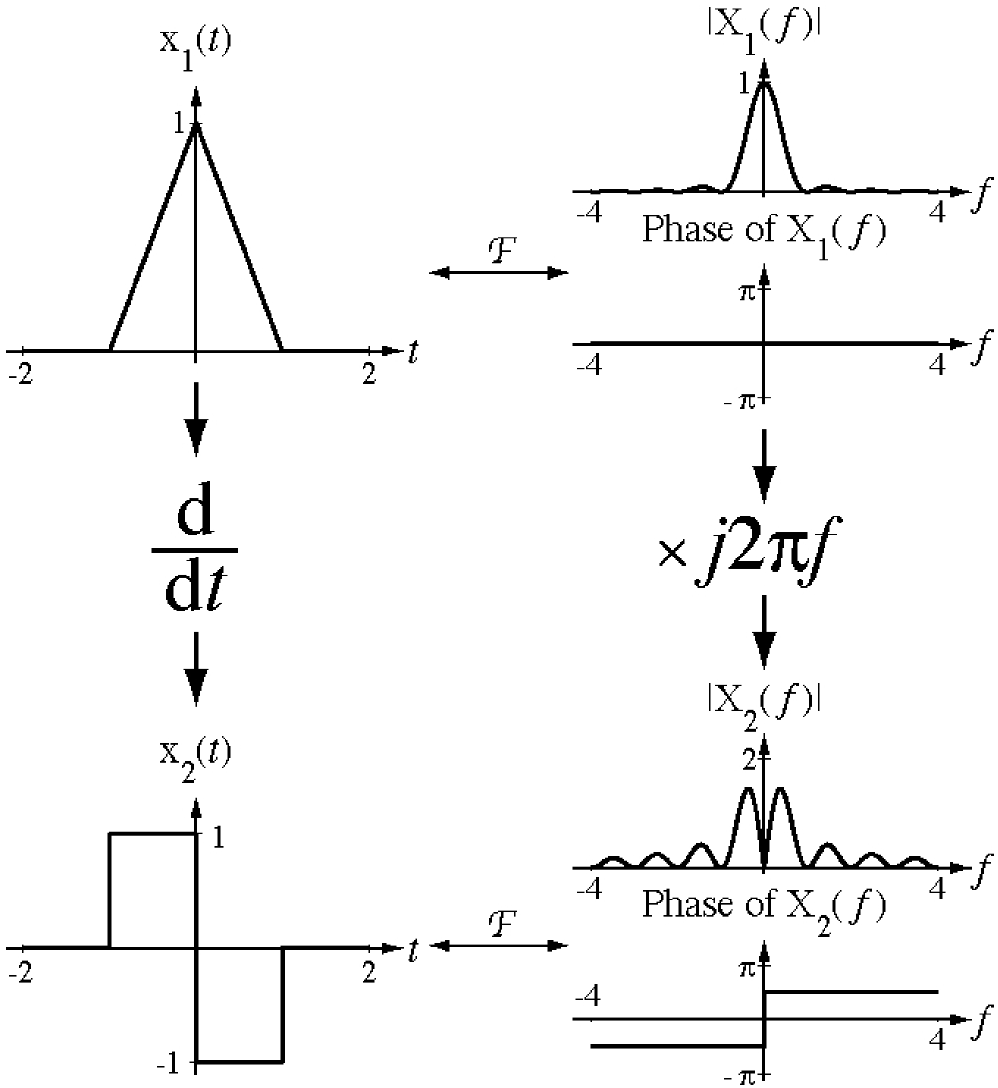
$$x(t)\cos(\omega_0 t) \overset{F}{\longleftrightarrow} \frac{1}{2}\left[X(j(\omega - \omega_0)) + X(j(\omega + \omega_0))\right]$$

Transforms of Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{-j2\pi(kf_F)t} \overset{F}{\longleftrightarrow} X(f) = \sum_{k=-\infty}^{\infty} X[k]\delta(f - kf_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{-j(k\omega_F)t} \overset{F}{\longleftrightarrow} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

CTFT Properties



CTFT Properties

Parseval's Theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 df$$

Integral Definition of an Impulse

$$\int_{-\infty}^{\infty} e^{-j2\pi xy} dy = \delta(x)$$

Duality

$$X(t) \xleftrightarrow{\mathrm{F}} x(-f) \text{ and } X(-t) \xleftrightarrow{\mathrm{F}} x(f)$$

$$X(jt) \xleftrightarrow{\mathrm{F}} 2\pi x(-\omega) \text{ and } X(-jt) \xleftrightarrow{\mathrm{F}} 2\pi x(\omega)$$

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CTFT Properties

Total-Area Integral

$$X(0)=\left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt\right]_{f\to 0}=\int_{-\infty}^{\infty} x(t)dt$$

$$x(0)=\left[\int_{-\infty}^{\infty} X(f)e^{+j2\pi ft}df\right]_{t\to 0}=\int_{-\infty}^{\infty} X(f)df$$

$$X(0)=\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]_{\omega\to 0}=\int_{-\infty}^{\infty} x(t)dt$$

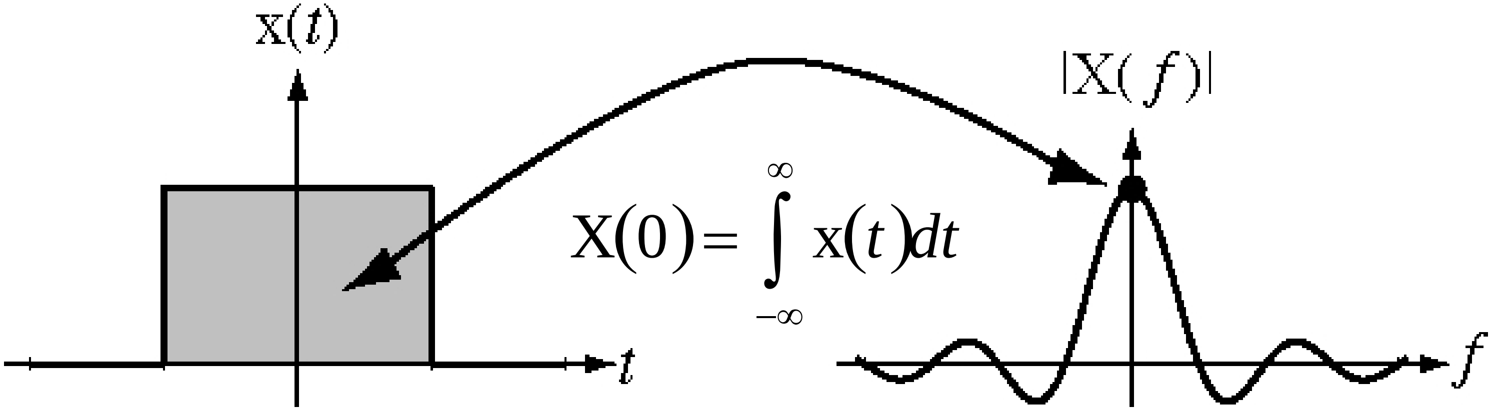
$$x(0)=\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{+j\omega t}d\omega\right]_{t\to 0}=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$$

Integration

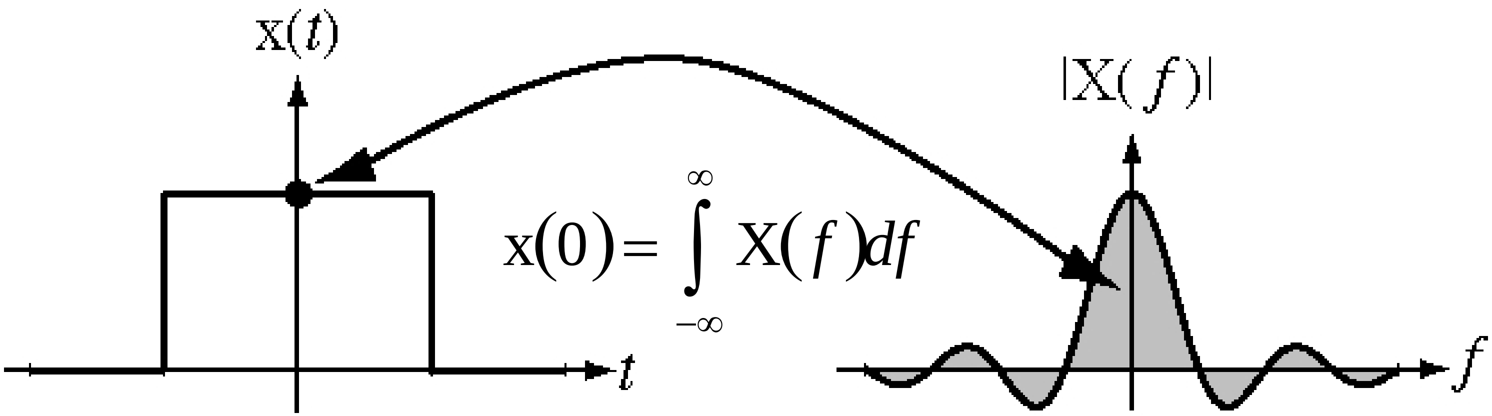
$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{\mathcal{F}} \frac{X(f)}{j2\pi f}+\frac{1}{2}X(0)\delta(f)$$

$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{\mathcal{F}} \frac{X(j\omega)}{j\omega}+\pi X(0)\delta(\omega)$$

CTFT Properties

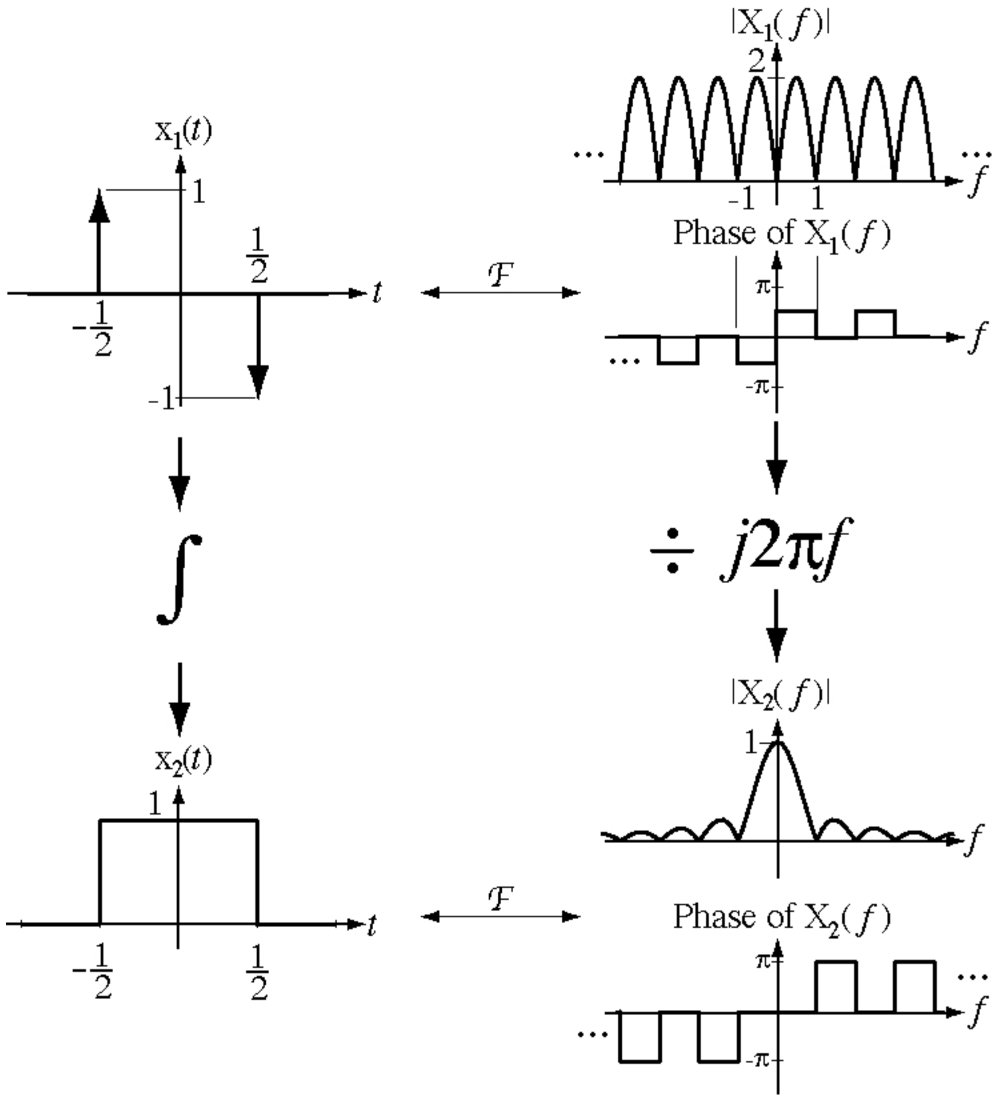


$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$



$$x(0) = \int_{-\infty}^{\infty} X(f)\,df$$

CTFT Properties



Example 1

If $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)$, what are the FTs of $x(2(t-1))$ and $x(2t-1)$?

Example 2

Find the FT of $10\sin(t) * 2\delta(t+4)$, using two different approaches:

- Approach 1: Find the convolution first and then the FT:

- Approach 2: Use the convolution property of FT:

Example 3

- Find the FT of $\delta(t)$ directly (do not use the table).

- Now find the FT of

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Example 4

What is the Inverse FT (IFT) of $e^{-4|f|}$?

Example5

We have previously seen that the impulse response of an RC lowpass filter is $h(t) = e^{-t/\tau}u(t)$. If the input voltage to this filter is $v_i(t) = e^{-bt}u(t)$ ($b > 0$), find the output voltage $v_o(t)$. Use the convolution property of FT.