ENSC-380, Spring 2009

Review Problems

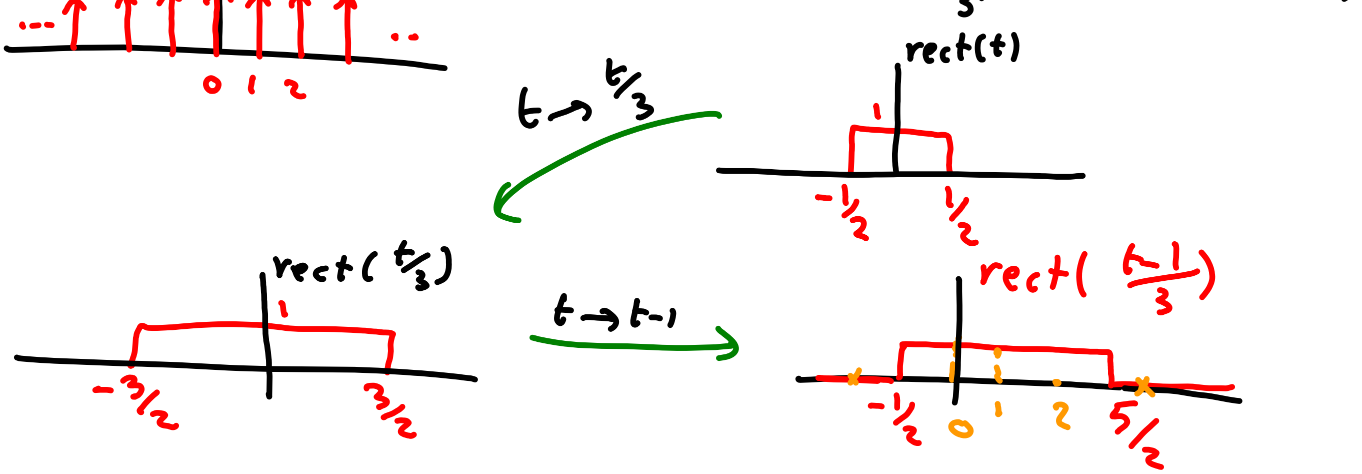
1. (Midterm,2008)

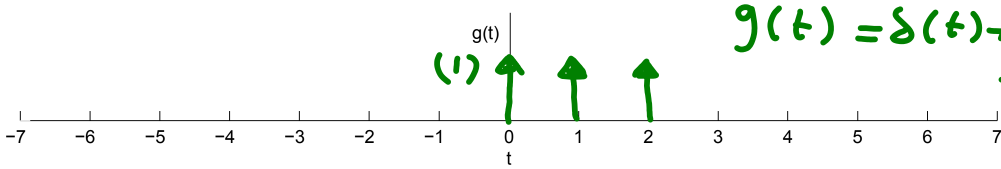
- **a)** Sketch the following signal:

$$g(t) = \mathrm{comb}(t).\mathrm{rect}(\frac{t-1}{3})$$

Use the space below for your intermediate work but show your final answer on the provided axes. Clearly indicate the values on both axes.

$$\mathrm{rect}(t) \xrightarrow{t \to t/3} \mathrm{rect}(t/3) \xrightarrow{t \to t-1} \mathrm{rect}(\frac{t-1}{3})$$





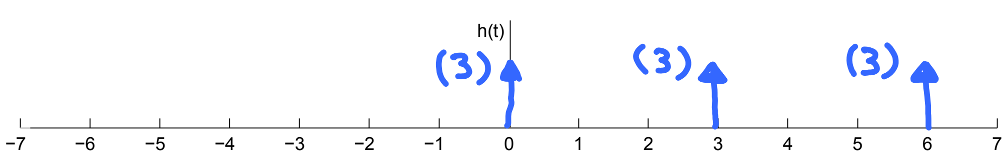
$$g(t) = \delta(t) + \delta(t-1) + \delta(t-2)$$

- **b)** Now sketch $h(t) = g(t/3)$

$$g(t) = \delta(t) + \delta(t-1) + \delta(t-2) \Longrightarrow h(t) = g(\tfrac{t}{3}) = \delta(\tfrac{t}{3}) + \delta(\tfrac{t}{3}-1) + \delta(\tfrac{t}{3}-2)$$

$$\Longrightarrow h(t) = \delta(\tfrac{t}{3}) + \delta(\tfrac{t-3}{3}) + \delta(\tfrac{t-6}{3})$$

$$= 3\delta(t) + 3\delta(t-3) + 3\delta(t-6)$$



2. (Midterm 2008) A CT-LTI system is defined with the following differential equation:

$$y'(t) + 2y(t) = x(t) + 2x'(t)$$

Find the impulse response of this system, $h(t)$. $h'(t) + 2h(t) = \delta(t) + 2\delta'(t)$

1) $t<0 \Rightarrow h(t) = 0$

2) $t>0 \Rightarrow h'(t) + 2h(t) = 0 \Longrightarrow h(t) = k_1 e^{\alpha t}$

$\cancel{k}\alpha \cancel{e^{\alpha t}} + 2\cancel{k}\cancel{e^{\alpha t}} = 0 \Rightarrow \alpha = -2 \Rightarrow h(t) = k_1 e^{-2t}$

3) Is $h(t) = k_1 e^{-2t} u(t)$ enough? No

$\Rightarrow h(t) = k_1 e^{-2t} u(t) + k_2 \delta(t)$

replace in diff. equ. for $t=0$:

$-2k_1 \cancel{e^{-2t} u(t)} + k_1 e^{-2t} \delta(t) + k_2 \delta'(t)$

$+ 2k_1 \cancel{e^{-2t} u(t)} + 2k_2 \delta(t) = \delta(t) + 2\delta'(t)$

at $t=0$

$\begin{cases} k_1 + 2k_2 = 1 \\ k_2 = 2 \end{cases} \Longrightarrow \begin{cases} k_1 = -3 \\ k_2 = 2 \end{cases} \Longrightarrow h(t) = -3e^{-2t}u(t) + 2\delta(t)$

3. (Midterm 2008) Use the properties of CTFS and the FS formula given below ()

$$\cos(2\pi f_0 t) \xleftrightarrow{FS} \frac{1}{2}\{\delta[k-1]+\delta[k+1]\} \qquad \textbf{Representation Period: } T_0 = \frac{1}{f_0}$$

to find the CTFS representation(Harmonic function) of

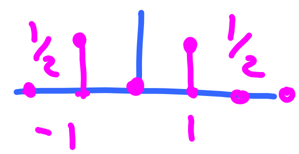
$$x(t) = \cos(200\pi t + \pi/4)$$

with **representation period of** $T_f = 1/50$.

Notes: 1) Do not use any other entries from the FS table. 2) Try to simplify your result as much as possible.

$$x(t) = \cos\left(2\pi \times 100 \times t + \frac{\pi}{4}\right) = \cos\left(2\pi \times 100\left(t + \frac{1}{800}\right)\right)$$

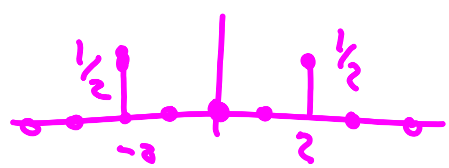
$$\cos(2\pi \times 100 \times t) \xleftrightarrow[T_f = T_0 = 1/100 \text{ or } f_f = 100]{FS} \underbrace{\frac{1}{2}\{\delta[k-1]+\delta[k+1]\}}_{Y[k]}$$



Change of Rep. period:

$$\cos(2\pi \times 100 \times t) \xleftrightarrow[T_f = 2T_0 = 1/50]{} \begin{cases} Y[k/2] & k/2 \text{ integ.} \\ 0 & \text{else} \end{cases}$$

$$\cos(2\pi 100 t) \xleftrightarrow[T_f = 1/50]{FS} \frac{1}{2}\{\delta[k-2]+\delta[k+2]\}$$



Time shift:

$$\cos\left(2\pi 100\left(t + \frac{1}{800}\right)\right) \xleftrightarrow[T_f = 1/50,\ f_f = 50]{FS}$$

$$X[k] = e^{+j2\pi k \times 50 \times \frac{1}{800}} \times \frac{1}{2}\{\delta[k-2]+\delta[k+2]\}$$

$$X[k] = e^{j\frac{2\pi \times 2 \times 50}{800}} \times \frac{1}{2}\delta[k-2]$$

$$+ e^{-j\frac{2\pi \times 2 \times 50}{800}} \times \frac{1}{2}\delta[k+2]$$

$$= \frac{1}{2}e^{j\frac{\pi}{4}}\delta[k-2] + \frac{1}{2}e^{-j\frac{\pi}{4}}\delta[k+2]$$

$$\delta(at) = \frac{1}{a}\delta(t)$$

but for DT: $\delta\left[\frac{n}{m}\right] = \delta[n]$

4. Text, Problem 4-25 part (b)
5. Text, Problem 4-29
6. Prove the "Time Reversal" property of CTFS.

4.25 (b) Find the CTFS of

$$x(t) = 5[\mathrm{tri}(t-1) - \mathrm{tri}(t+1)] * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right), \quad T_F = 4$$

4.29 Find the signal power of $x[n]$:

$$x[n] = 5\sin\left(\frac{14\pi n}{15}\right) - 8\cos\left(\frac{26\pi n}{30}\right)$$

$$\left.\begin{aligned} x_1[n] &= \sin\left(\frac{14\pi n}{15}\right) \Rightarrow N_1 = 15 \\ x_2[n] &= \cos\left(\frac{2\pi\times 13\times n}{30}\right) \Rightarrow N_2 = 30 \end{aligned}\right\} \Rightarrow N_f = 30$$

Find the smallest intg. N_1 s.t.

$$\sin\left(\frac{14\pi(n+N_1)}{15}\right) = \sin\left(\frac{14\pi n}{15}\right)$$

$$\frac{\cancel{14}^{\,7}\pi N_1}{15} = k\times\cancel{2\pi} \Rightarrow 7N_1 = k\times 15$$

$N_1 = 15$ $\quad 7$

from Appendix E:

$$\cos\left(\frac{2\pi n}{N_0}\right) \xleftrightarrow{N_f = mN_0} \frac{1}{2}\left(\text{comb}_{N_f}[k-m] + \text{comb}_{N_f}[k+m]\right)$$

$$\sin\left(\frac{2\pi n}{N_0}\right) \xleftrightarrow{N_f = mN_0} \frac{j}{2}\left(\text{comb}_{N_f}[k+m] - \text{comb}_{N_f}[k-m]\right)$$

N_0 is not necessarily an integer!

$$x_1[n] = \sin\left(\frac{14\pi n}{15}\right) = \sin\left(\frac{2\pi n}{15/7}\right)$$

$$N_0 = \frac{15}{7}$$

$$\sin\left(\frac{2\pi n}{15/7}\right) \underset{N_f = 30 = \underbrace{14}_{m} N_0}{\overset{FS}{\longleftrightarrow}} \frac{j}{2}\left\{ \mathrm{Comb}_{30}[k+14] - \mathrm{Comb}_{30}[k-14] \right\}$$

$$x_2[n] = \cos\left(\frac{26\pi n}{30}\right) = \cos\left(\frac{2\pi n}{\underbrace{30/13}_{N_0}}\right)$$

$$x_2[n] \underset{N_f = 30 = \underbrace{13}_{m} N_0}{\overset{FS}{\longleftrightarrow}} \frac{1}{2}\left\{ \mathrm{Comb}_{30}[k-13] + \mathrm{Comb}_{30}[k+13] \right\}$$

$$\vdots$$

Prove the "time reversal" property for CTFS.

(1) $x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j2\pi k f_0 t}$

suppose $y(t) = x(-t) = \sum_{k=-\infty}^{\infty} Y[k]\, e^{j2\pi k f_0 t}$ (2)

(1) $\Rightarrow x(-t) = \sum_{k} X[k]\, e^{-j2\pi k f_0 t}$

$k \to -k \quad x(-t) = \sum_{k=-\infty}^{\infty} X[-k]\, e^{j2\pi k f_0 t}$ (3)

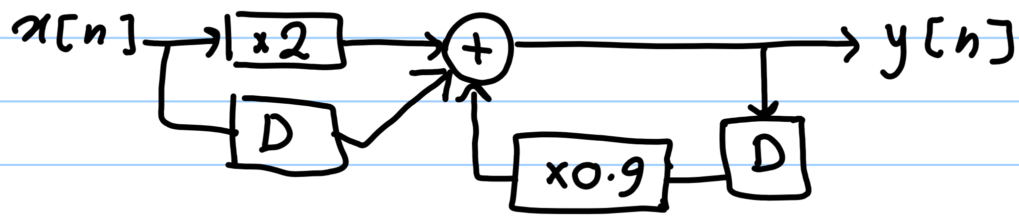
compare (2) & (3)

$$\Rightarrow Y[k] = X[-k]$$

Thus if $x(t) \xleftrightarrow{FS} X[k]$

then $x(-t) \xleftrightarrow{FS} X[-k]$

3.43 Find the impulse response of this system:



$$y[n] = 0.9\,y[n-1] + 2x[n] + x[n-1]$$

$$\Rightarrow y[n] - 0.9\,y[n-1] = 2x[n] + x[n-1] \qquad (*)$$

Assume $h_1[n]$ to be the impulse response to the following system:

$$y[n] - 0.9\,y[n-1] = 2x[n]$$

Also assme that $h_2[n]$ is the impl. resp. to another system:

$$y[n] - 0.9\,y[n-1] = x[n-1]$$

Then the impulse response to the original system $(*)$ is $h[n] = h_1[n] + h_2[n]$

On the other hand, $h_2[n] = \frac{1}{2} h_1[n-1]$

So it is enough to find $h_1[n]$:

$$h_1[n] - 0.9\,h_1[n-1] = 2\,\delta[n]$$

- For $n<0 \Rightarrow h_1[n]=0$

- For $n>0 \Rightarrow h_1[n]=k\,(0.9)^n$

- at $n=0 \Rightarrow h_1[0]-0.9\underbrace{h_1[-1]}_{0}=2$

$\Rightarrow h_1[0]=2 \Rightarrow k=2 \Rightarrow h_1[n]=2\,(0.9)^n u[n]$

Now, $h_2[n]=\frac{1}{2}h_1[n-1]$

$\Rightarrow h_2[n]=(0.9)^{n-1}u[n-1]$

Finally: $h[n]=h_1[n]+h_2[n]$

$=2\times(0.9)^n u[n]+(0.9)^{n-1}u[n-1]$

$=2\times(0.9)^0\delta[n]+2\times(0.9)^n u[n-1]+(0.9)^{n-1}u[n-1]$

$=2\delta[n]+(0.9)^{n-1}u[n-1]\,(1.8+1)$

$=2\delta[n]+2.8\times(0.9)^{n-1}u[n-1]$