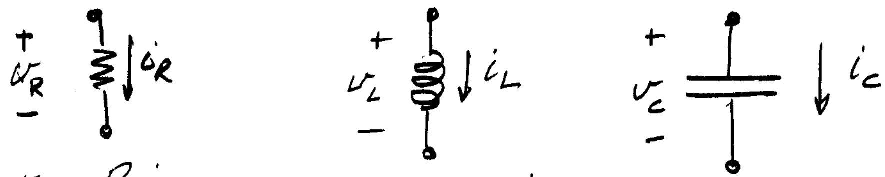
1.2 Analysis of Linear Circuits by Differential Equations

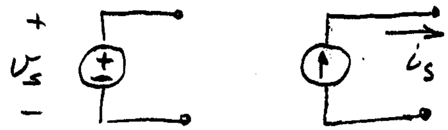
DC & L Chapt. 9

- The DE representation of a circuit is fundamental. It accommodates time variation and non-linearities, both of which present problems for transform methods.

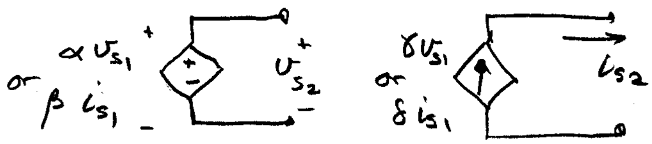
- The cast:



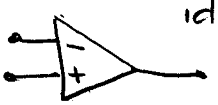
$v_R = R\,i_R$ $\qquad$ $v_L = L\dfrac{di_L}{dt}$ $\qquad$ $i_C = C\dfrac{dv_C}{dt}$



independent sources



controlled sources



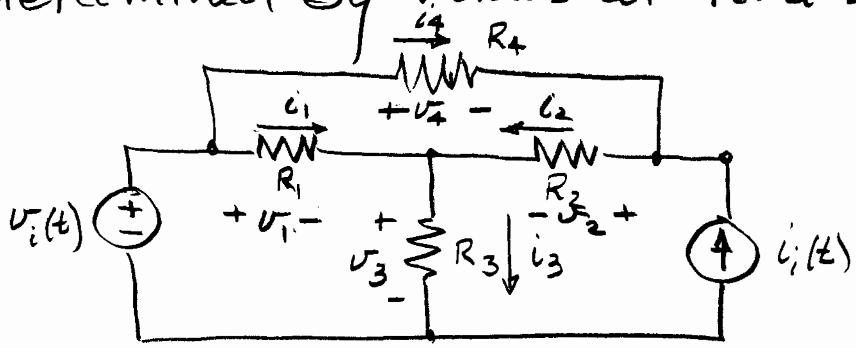
ideal op amp

Why are these idealizations?

- From previous courses, you are familiar with use of node and loop equations (KCL, KVL) to obtain a DE relating a selected variable as output to the various inputs. We won't expand on the method in ENSC 320 — but you must know it.

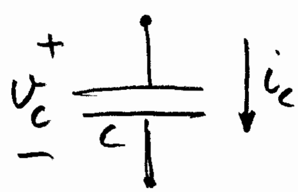
- Instead, we will develop an alternative method based on "state equations."
  - convenient for numerical solution by computer
  - convenient for non linear circuits
  - link to control theory, system theory
  - link to linear algebra.

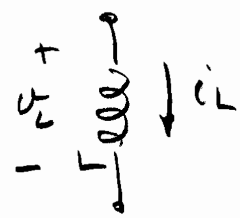
- In a purely resistive circuit, the values at time t of every current and voltage are determined by values at time t of the inputs



memoryless

- But capacitors and inductors have memory:





$$i_c = C \frac{dv_c}{dt}$$

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(\alpha)\,d\alpha$$

cap voltage won't change instantaneously, even if sources do.

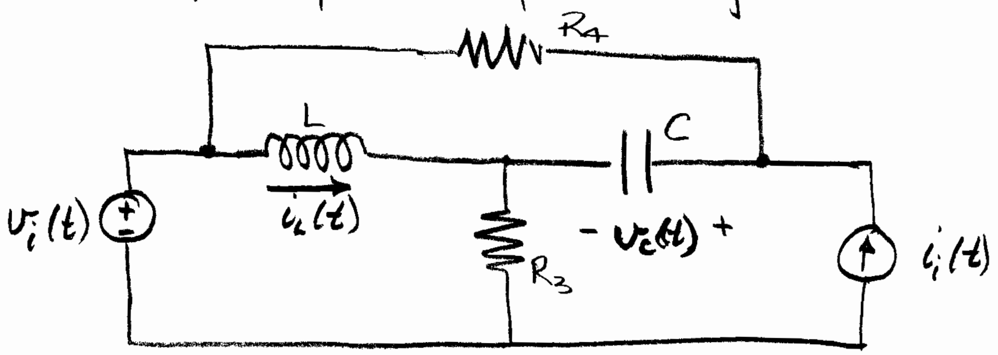
$$v_L = L\frac{di_L}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\alpha)\,d\alpha$$

inductor current won't change instantaneously, even if sources do

Capacitor voltages and inductor currents summarize the effect at time t of all past inputs.

- In an RLC circuit, the values at time t of every current and voltage are determined by the values at time t of the inputs, the cap voltages and inductor currents.



With inputs, cap voltages, inductor currents specified, you can calculate all other currents and voltages by loop & node equations, or circuit reduction (voltage dividers, current dividers, etc), or (for simple circuits) by inspection.

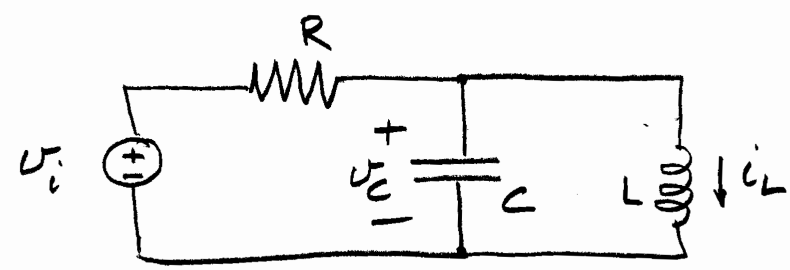
- So the response of the circuit is determined by the way the v_c's and i_L's respond; we need their derivatives $\frac{d}{dt} v_c(t)$, $\frac{d}{dt} i_L(t)$.

- The state equations express the derivatives of the cap voltages and inductor currents (the state variables) as functions <u>only</u> of
  - the inputs
  - the state variables themselves.

not (e.g.) resistor voltages, or cap currents, or inductor voltages.

"Coupled 1st order DEs."

<u>Example</u>



state variables $v_c(t)$, $i_L(t)$.

State equations:

$$\dot{v}_c = \frac{1}{C} i_c = \frac{1}{C}\left(\frac{v_i - v_c}{R} - i_L\right) \qquad \text{KCL}$$

$$= -\frac{1}{RC} v_c - \frac{i_L}{C} + \frac{v_i}{RC} \qquad (1)$$

and

$$\dot{i}_L = \frac{1}{L} v_L = \frac{1}{L} v_c \qquad (2) \qquad \text{KVL}$$

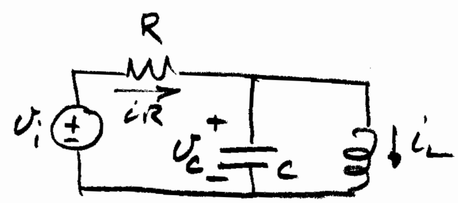
The v_c and i_L act rather like sources themselves in determing their derivatives.

- Link with linear algebra and with numerical solution.

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} v_i$$

- If we solve for $v_c(t)$ and $i_L(t)$, then other circuit quantities are determined — for example, $i_R = \frac{v_i - v_c}{R}$

$$= \begin{bmatrix} -\frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \frac{1}{R} v_i$$



- General approach for state equations;
  1. Write elementary DE for each state var.

     $\dot{v}_c = \frac{1}{C} i_c \qquad \dot{i}_L = \frac{1}{L} v_L$

  2. Express RHS (i_c, v_L) in terms of inputs and state vars v_c, i_L <u>only</u>.

     Note i_c, v_L are <u>not</u> state variables.

- Often, we are interested in one quantity, which we designate as an "output". We want a single higher order DE for it, instead of a collection of 1st order DEs.
  - Suppose it's v_c, in example from p. 1.2.2.d.
  - start from state eq'ns

$$\dot{v}_c = -\frac{1}{RC} v_c - \frac{1}{C} i_L + \frac{1}{RC} v_i \qquad (1)$$

$$\dot{i}_L = \frac{1}{L} v_L \qquad (2)$$

  - Eliminate i_L by differentiating (1), subst (2):

$$\ddot{v}_c = -\frac{1}{RC}\dot{v}_c - \frac{1}{C}\dot{i}_L + \frac{1}{RC}\dot{v}_i \qquad \dot{(1)}$$

$$= -\frac{1}{RC}\dot{v}_c - \frac{1}{LC} v_c + \frac{1}{RC}\dot{v}_i \qquad \text{subst (2)}$$

Rearrange:

$$\ddot{v}_c + \frac{1}{RC}\dot{v}_c + \frac{1}{LC} v_c = \frac{1}{RC}\dot{v}_i$$

$$\boxed{\text{General procedure DC \& L 9.5}}$$

- It's a bit more complicated to obtain a DE in i_L, since ① has v_C on the right hand side. This way works well enough, since it's a simple circuit:

$$\ddot{i}_L = \frac{1}{L}\dot{v}_C$$

$$= -\frac{1}{RLC}v_C - \frac{i_L}{LC} + \frac{v_i}{RLC} \quad \text{subst ①}$$

$$= -\frac{\dot{i}_L}{RC} - \frac{i_L}{LC} + \frac{v_i}{RLC} \quad \text{subst ②}$$

$$\boxed{\ddot{i}_L + \frac{1}{RC}\dot{i}_L + \frac{1}{LC}i_L = \frac{1}{RLC}v_i} \quad \text{rearranged}$$

- The systematic procedure of DC&L 9.5 treats differentiation like mult. by a constant. OK, since differentiation operator and "mult by a const" operator are linear and commute with themselves and each other.

$$\left(\frac{d}{dt} + \frac{1}{RC}\right)v_C + \frac{1}{C}i_L = \frac{1}{RC}v_i \quad \text{① (rewritten)}$$

$$\frac{1}{L}v_C - \frac{d}{dt}i_L = 0 \quad \text{② (rewritten)}$$

Eliminate v_C:

first, $L\left(\frac{d}{dt} + \frac{1}{RC}\right)$②: $\left(\frac{d}{dt} + \frac{1}{RC}\right)v_C - L\left(\frac{d}{dt} + \frac{1}{RC}\right)\frac{d}{dt}i_L = 0 \quad \hat{②}$

second, ①−$\hat{②}$: $\frac{1}{C}i_L + L\left(\frac{d}{dt} + \frac{1}{RC}\right)\frac{d}{dt}i_L = \frac{1}{RC}v_i \quad ③$

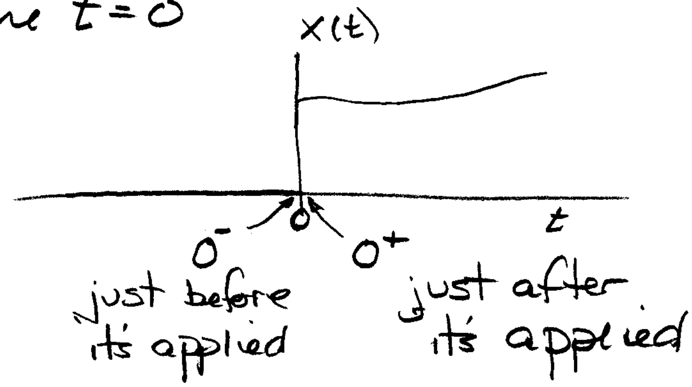
$\frac{1}{L}$③, then rearrange:

$$\boxed{\ddot{i}_L + \frac{1}{RC}\dot{i}_L + \frac{1}{LC}i_L = \frac{1}{RLC}v_i}$$

This is easier with Laplace transforms.

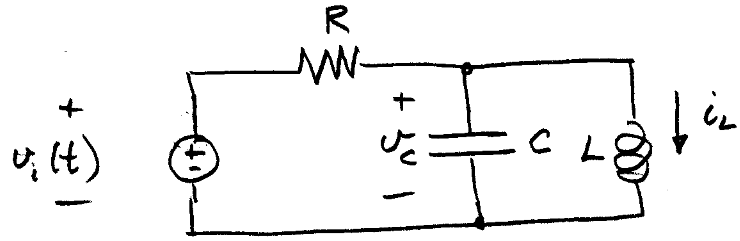
- Response depends on
  - the input(s)
  - what the circuit was doing just before the input was applied (i.e., cap voltages, inductor currents.

- Consider inputs that are switched on at time $t=0$



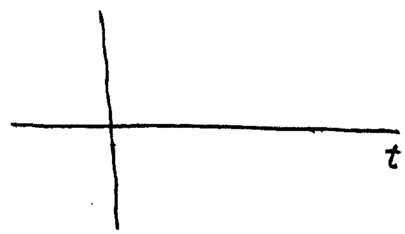
This notation lets us distinguish initial conditions ($t=0^-$) from those just after input starts ($t=0^+$). Lets us account for discontinuities by transforming 0^- conditions to 0^+ conditions, the ones we use for fitting parameters.

- Example – series/parallel RLC again



Suppose $v_c(0^-) = 0$ $\quad i_L(0^-) = 0$ $\quad$ at rest

and $v_i(t) = \begin{cases} V_i \cos(2\pi f_o t) & t \geq 0 \\ 0 & t < 0 \end{cases}$



What are $v_c(0^+)$, $i_L(0^+)$?

what are $\dot{v}_c(0^+)$, $\dot{i}_L(0^+)$?

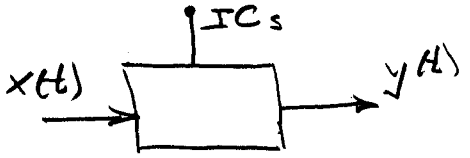
$$\dot{v}_c(0^+) = \frac{1}{C} i_c(0^+) = \frac{1}{C}\left(\frac{v_i(0^+) - v_c(0^+)}{R} - i_L(0^+)\right)$$

$$=$$

$$\dot{i}_L(0^+) = \frac{1}{L} v_L(0^+) = \frac{1}{L} v_c(0^+) =$$

$\boxed{\text{DC \& L Ex 9.8}}$

- Generally, the response is the solution to



$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = b_m x^{(m)} + \cdots + b_0 x \quad (m \leq n)$$

with ICs $\quad y^{(k)}(0^-) = c_k$

- Components of the solution
  - A particular solution satisfies the DE
    $$A[y_p(t)] = B[x(t)]$$
    for the given $x(t)$, but does not necessarily satisfy the ICs.
  - A homogeneous solution satisfies
    $$A[y_h(t)] = 0$$
    but not necessarily the ICs. It depends only on the system internal dynamics, not the input.
  - The sum of any particular and any homog. solution satisfies the DE, so total solution
    $$y_t(t) = y_p(t) + y_h(t)$$
    since
    $$A[y_t(t)] = A[y_p(t)] + A[y_h(t)]$$
    $$= B[x(t)]$$

– Reminiscent of linear equations

$$A \underline{y} = \underline{b}$$

If A is singular, then $A \underline{y}_h = \underline{0}$ where $\underline{y}_h$ has number of params equal to degeneracy.

If $\underline{y}_p$ is any solution of $A \underline{y}_p = \underline{b}$, then so is $\underline{y}_p + \underline{y}_h$ since

- From MATH 310

Homogeneous

Exponentials reproduce under differentiation, so substitute

$$y_h(t) = K e^{st}$$

giving

$$\underbrace{(a_n s^n + a_{n-1} s^{n-1} + \dots a_0)}_{=0} K e^{st} = 0$$

Get n solutions s_i, $i = 1..n$ as real, or complex pairs and possibly repeated.

Hence

$$y_h(t) = \sum_{i=1}^{n} K_i e^{s_i t}$$

$$= K_1 e^{s_1 t} + K_2 e^{s_2 t} + \dots + K_n e^{s_n t}$$

is the general form. Repeated roots give

$$\text{poly}(t)\, e^{s_i t}$$

where poly(t) degree is one less than multiplicity of root.

Particular

From experience

$x(t)$	$y_p(t)$
X	Y (consts)
$X_0 + X_1 t$	$Y_0 + Y_1 t$
$\cos(2\pi f_0 t)$ or $\sin(2\pi f_0 t)$	$Y_c \cos(2\pi f_0 t) + Y_s \sin(2\pi f_0 t)$

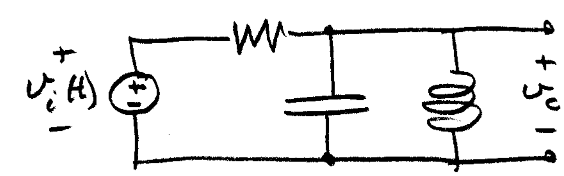
Substitute $y_p(t)$ into DE and fit parameters so both sides match

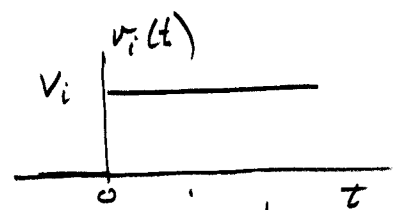
- The total solution

$$y_t(t) = y_p(t) + \underbrace{y_h(t)}_{n \text{ params}}$$

must now be adjusted through choice of params to match the ICs. Fit them at what time??

Note $y_t(t)$ is continuous for $t > 0$, so fit at $t = 0^+$ to the conditions we transformed from those at $t = 0^-$.

- Example 



step input

Particular $v_{cp}(t) = V_c$ since input constant

$$\ddot{V}_c + \frac{1}{RC}\dot{V}_c + \frac{1}{LC}V_c = \underbrace{\frac{1}{RC}\dot{V}_i}_{0}$$

so $v_{cp}(t) = 0$ on physical grounds, too

Homogeneous — find s values. Subst e^{st} into DE p. 1.2.2

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad s = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}$$

$$v_{ch}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

Total

$$v_c(t) = v_{cp}(t) + v_{ch}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \quad t > 0$$

And fit to ICs on p. 1.2.4.

$$v_c(0^+) = K_1 e^{s_1 0^+} + K_2 e^{s_2 0^+} = 0 \qquad K_1 + K_2 = 0$$

$$\dot{v}_c(0^+) = K_1 s_1 e^{s_1 0^+} + K_2 s_2 e^{s_2 0^+} = \frac{V_i}{RC} \qquad K_1 s_1 + K_2 s_2 = \frac{V_1}{RC}$$

so $$K_1 = \frac{V_i}{RC\,(s_1 - s_2)} \qquad K_2 = -K_1$$

or $$= \frac{V_i}{RC\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}} \qquad K_2 = -K_1$$

where $s_1 = -\frac{1}{2RC} + \frac{1}{2}\sqrt{\cdots}$

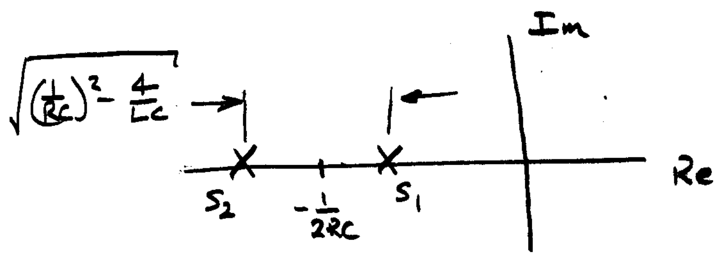
$s_2 = -\frac{1}{2RC} - \frac{1}{2}\sqrt{\cdots}$

so $$v_c(t) = \frac{V_i}{RC\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}\, e^{-t/2RC}\left(e^{\frac{t}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}} - e^{-\frac{t}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}\right) \quad (1)$$

Looks as though we have a complete solution method. But what if the discriminant is negative (complex roots) or zero (repeated roots)?

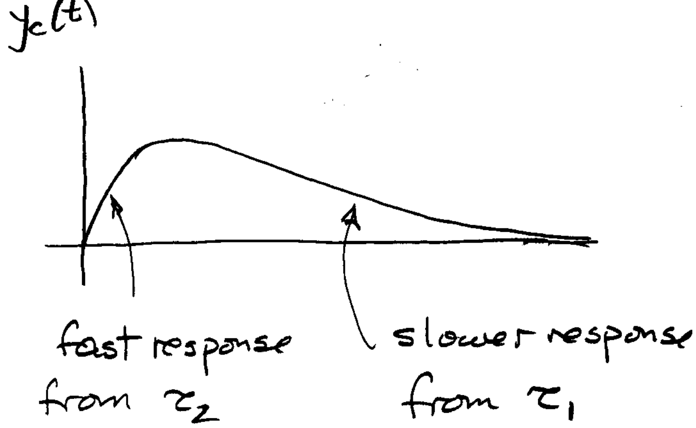
A little more work is required (not much). Look at all three cases in detail: distinct real roots, repeated real roots, imaginary roots.

– Case 1: $\left(\frac{1}{RC}\right)^2 > \frac{4}{LC}$, distinct real roots



This is the solution labelled 1 on p. 1.2.8. View it as $e^{-t/2RC}$ decaying exponential times a sinh function or, more productively, as

$$v_c(t) = \frac{V_i}{RC\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{1}{LC}}}\left(\underbrace{e^{t\left(-\frac{1}{2RC}+\frac{1}{2}\sqrt{\cdots}\right)} - e^{t\left(-\frac{1}{2RC}-\frac{1}{2}\sqrt{\cdots}\right)}}_{e^{-t/\tau_1} - e^{-t/\tau_2}}\right)$$



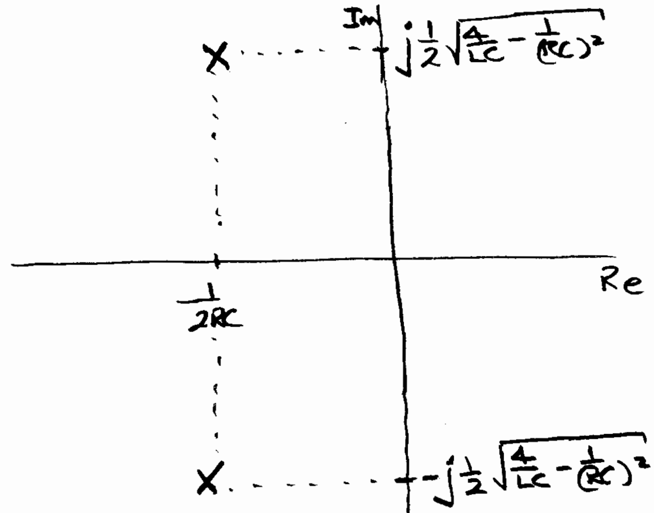
$\tau_1 =$

$\tau_2 =$

The root closer to the origin contributes the more slowly decaying term. Generally true – real poles closer to origin have slower responses.

– Case 2: $\left(\frac{1}{RC}\right)^2 < \frac{4}{LC}$, complex roots

$$s_1 = -\frac{1}{2RC} + \frac{j}{2}\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}} \qquad s_2 = -\frac{1}{2RC} - \frac{j}{2}\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}}$$

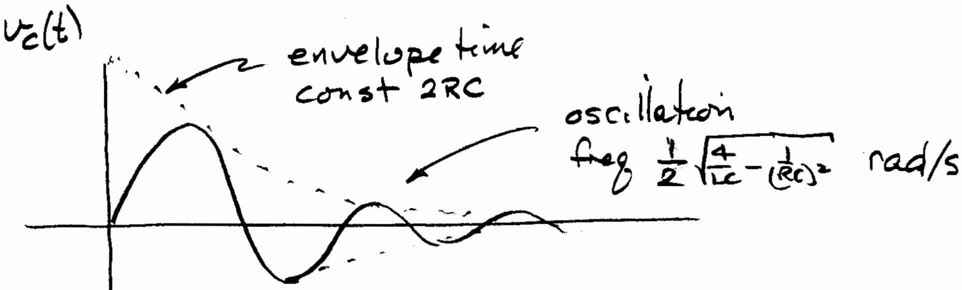


a conjugate pair

From ①, p 1.2.8

$$v_c(t) = \frac{V_i}{jRC\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}}}\, e^{-t/2RC} \underbrace{\left(e^{j\frac{t}{2}\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}}} - e^{-j\frac{t}{2}\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}}} \right)}_{2j\sin\left(\frac{1}{2}\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}}\, t\right)}$$

$$= \frac{2V_i}{RC\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}}}\, e^{-t/2RC} \sin\left(\frac{1}{2}\sqrt{\frac{4}{LC} - \frac{1}{(RC)^2}}\, t\right)$$

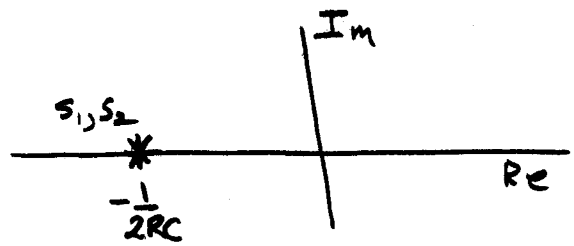


Imaginary part of roots farther from origin gives faster oscillation, closer to origin is slower osc.

Real part farther from origin gives faster decay, closer to origin, slower decay.

– Case 3 (last one) $\left(\frac{1}{RC}\right)^2 = \frac{4}{LC}$ repeated roots

$$s_1 = s_2 = -\frac{1}{2RC}$$



From ①, p 1.2.8, both numerator and denominator are zero! Two workarounds: (1) take limit of Case 1 or Case 2 as $s_1 - s_2 \to 0$ using L'Hopital's Rule, or (2) use the "canned" solution that results:

$$v_{ch}(t) = (F_1 + F_2 t)\, e^{s_c t} \qquad \text{(s_c is the common value of root)}$$

so $\quad v_c(t) = (F_1 + F_2 t)\, e^{s_c t}$

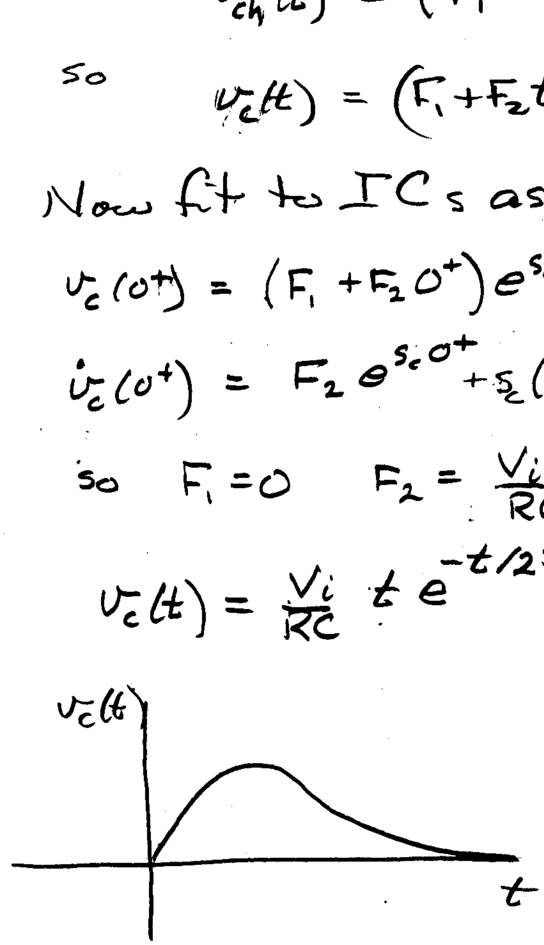
Now fit to ICs as on p 1.2.8

$$v_c(0^+) = (F_1 + F_2 0^+)\, e^{s_c 0^+} = 0 \Rightarrow F_1 = 0$$

$$\dot{v}_c(0^+) = F_2 e^{s_c 0^+} + s_c (F_1 + F_2 0^+) e^{s_c 0^+} = \frac{V_i}{RC} \Rightarrow s_c F_1 + F_2 = \frac{V_i}{RC}$$

so $\quad F_1 = 0 \quad F_2 = \frac{V_i}{RC} =$

$$v_c(t) = \frac{V_i}{RC}\, t\, e^{-t/2RC}$$



Faster response as pole pair moves out from origin.

- This lengthy example and its predecessors showed the full solution process:
  - obtain a single DE
  - convert ICs at 0^- to ICs at 0^+ for fitting parameters
  - the parameter fit
  - variations on the form of the response

It's easier with Laplace transforms!

- Why is this circuit useful?

View animations in "Poles, Zeros and System Response" on website.