1.3 Additional Topics on Differential Eqn Solution

- Section 1.2 described how to analyze simple electric circuits. Simple? No dependent sources, only one source. See DC&L for them.
- This section contains observations about the solution.

1.3.1 The Impulse Response

- We have seen how to calculate the response to a step, or a ramp, or a sinusoid. For ICs at $t=0^+$, we relied on continuity of cap voltages, inductor currents.
- Impulse response is tougher — its just a homogeneous solution for $t>0$ (Why?), but impulses can cause jumps in v_c or i_L. What ICs at $t=0^+$?

- How to obtain the impulse response:
  - the hard way – make subtle arguments about ICs before and after the impulse. See "How to Deal With Discontinuities" in Supplementary Notes and Demos on website.
  - the easy way – calculate the step response $g(t)$ and differentiate it:

$$h(t) = \frac{d}{dt} g(t)$$

1.3.2 Decomposition of Solution

- The response of a circuit can be split up in a few interesting ways
- You have seen

$$y_t(t) = y_p(t) + y_h(t)$$

with the n params of $y_h(t)$ selected so $y_t(t)$ fits ICs at $t = 0^+$

- Here's an important decomposition:
  - zero input response: response to initial conditions only, with no input; it's a homogeneous sol'n with params selected to satisfy ICs (spec'd at 0^-)
  - zero state response: response of system to input if it had been at rest; homog plus particular, with params selected for ICs all zero at 0^-.
  - total solution: $y_t(t) = y_{zs}(t) + y_{zi}(t)$

    Proof:

$$A[y_t] = A[y_{zs} + y_{zi}] = A[y_{zs}] + A[y_{zi}] = B[x], \quad t > 0$$

and

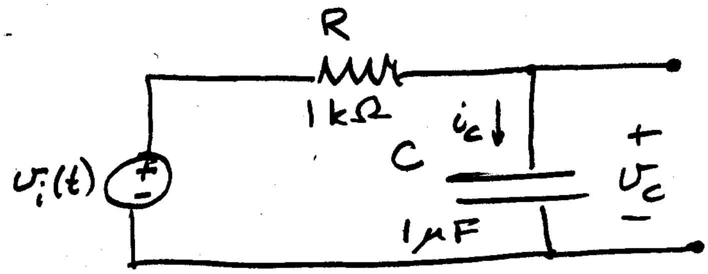
$$y_t^{(k)}(0^-) = y_{zs}^{(k)}(0^-) + y_{zi}^{(k)}(0^-) = 0 + c_k = c_k, \quad k = 0 .. n-1$$

so

$$A[y_t] = B[x], \quad t > 0$$

$$y_t^{(k)}(0^-) = c_k \qquad k = 0 .. n-1$$

- <u>example</u>: The circuit below has input, parameter values and IC's as shown. <u>Find</u> its voltage response as a sum of zero state and zero input responses, and verify that it satisfies the DE and ICs



$v_c(0^-) = 5$ volt.

$v_i(t) = 10\, u(t)$

- <u>The DE</u>

$$\dot{v}_c = \frac{1}{C} i_c = \frac{1}{C}\left(\frac{v_i - v_c}{R}\right)$$

$$\boxed{\dot{v}_c + \frac{1}{RC} v_c = \frac{1}{RC} v_i, \quad \text{with } v_c(0^-) = 5 \text{ volt}, \ v_i(t) = 10 u(t)}$$

- <u>zero input response</u> (due to ICs alone)

$$\dot{v}_{zir} + \frac{1}{RC} v_{zir} = 0 \Rightarrow v_{zir}(t) = \underbrace{A e^{-t/RC} = A e^{-1000t}}_{\text{a homogeneous solution}}$$

To get A, fit at $t=0^+$. With no input,

$$v_c(0^+) = v_c(0^-) = 5V; \quad v_{zir}(0^-) = A e^{-0^-/RC} = A$$

$A = 5V$

$$\boxed{v_{zir}(t) = 5 e^{-1000t}, \quad t > 0}$$

- <u>zero state response (circuit at rest, then driven by input)</u>

$$\dot{v}_{zsr}(t) + \frac{1}{RC} v_{zsr}(t) = \frac{10}{RC} u(t), \quad v_{zsr}(0^-) = 0 \ (\text{at rest})$$

particular solution for $t > 0$: $u(t)$ is constant at 1, so

$$v_p(t) = 10$$

homog solution, as ~~before~~,

$$v_h(t) = A e^{-t/RC} = A e^{-1000t}$$

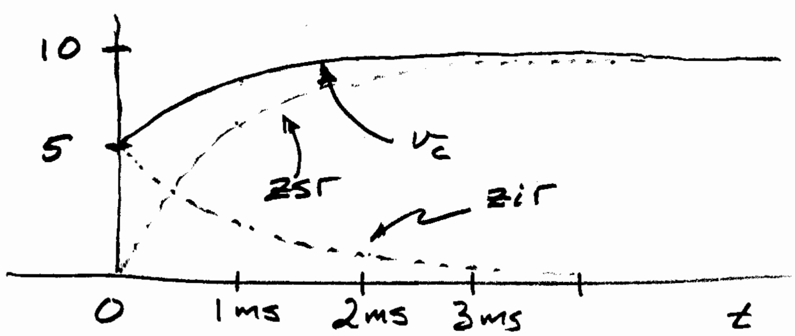
- So the zsr is $v_{zsr}(t) = 10 + Ae^{-1000t}, \quad t > 0$
- Transform the IC: No impulses driving circuit, so cap voltage is continuous; $v_{zsr}(0^+) = v_{zsr}(0^-) = 0$.

$$10 + Ae^{-1000 \cdot 0^+} = 0 \Rightarrow A = -10$$

$$\boxed{v_{zsr}(t) = 10(1 - e^{-1000t}), \quad t > 0}$$

- Total response

$$\boxed{\begin{aligned} v_c(t) &= \underbrace{5e^{-1000t}}_{zir} + \underbrace{10(1 - e^{-1000t})}_{zsr}, \quad t > 0 \\ &= 10 - 5e^{-1000t}, \quad t > 0 \end{aligned}}$$



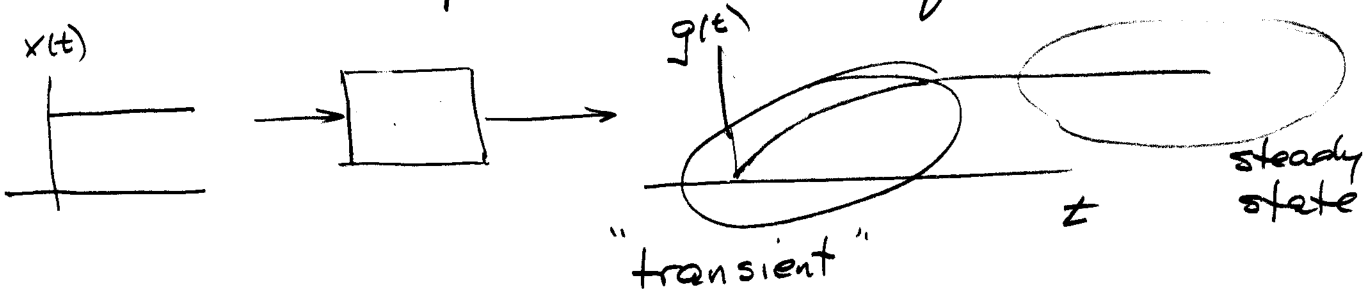
- Verify — substitute $v_c(t) = 10 - 5e^{-1000t}$ into the DE

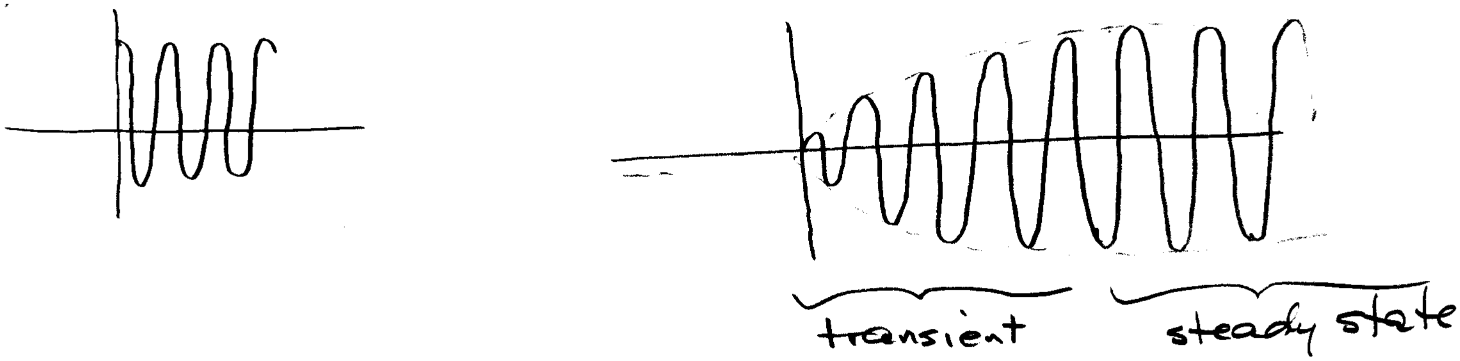
$$\underbrace{5000e^{-1000t}}_{\dot{v}_c} + \underbrace{1000}_{\frac{1}{RC}} \cdot \underbrace{(10 - 5e^{-1000t})}_{v_c} = \underbrace{1000}_{1/RC} \cdot \underbrace{10}_{V_i} \quad \checkmark$$

and IC?

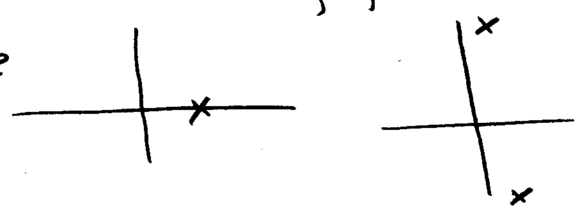
$$v_c(0^+) = 10 - 5e^{-1000 \cdot 0^+} = 5 \text{ V} \quad \checkmark$$

• Another decomposition is more qualitative.





- No sharp boundary
- Requires system to be stable, poles in LHP — not like

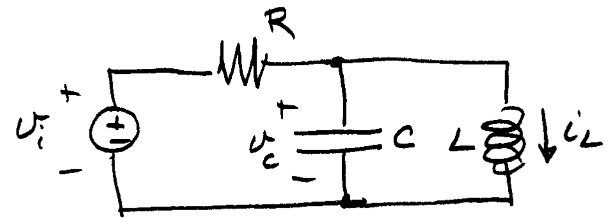


That way the homogeneous solution dies out (is transient) instead of growing exponentially.

- The steady state component is the particular solution.

1.3.3 State Variables

- Capacitors and inductors are the dynamic portions of a circuit. Their voltages v_c and currents i_L are "state variables".
- Recall initial analysis as coupled first order DEs. Example p 1.2.2

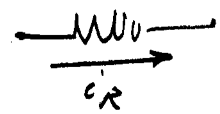


$$\dot{v}_c = -\frac{1}{RC} v_c - \frac{i_L}{C} + \frac{v_i}{RC}$$

$$\dot{i}_L = \frac{1}{L} v_c$$

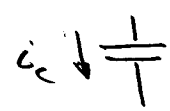
- Any selected output depends only on the input and the state variables.

  e.g. resistor current



$$i_R(t) = \frac{v_i(t) - v_c(t)}{R}$$

  e.g. cap current



$$i_c(t) = i_R(t) - i_L(t) = -\frac{1}{R} v_c(t) - i_L(t) + \frac{1}{R} v_i(t)$$

- The evolution of the system response depends only on the input and the state variables, since the derivatives capture the instantaneous evolution

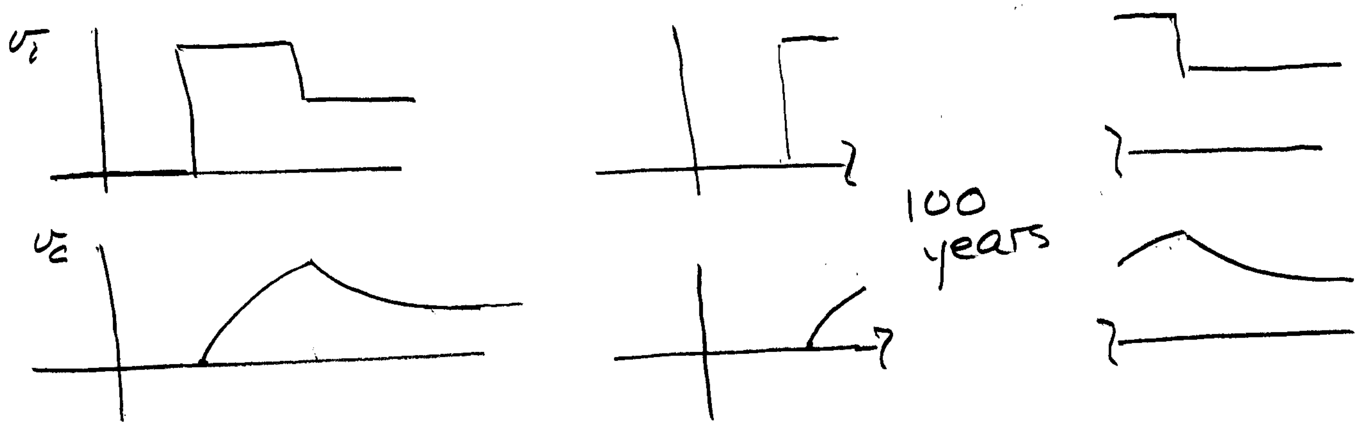
- The state (the collection of state variables) captures the entire past history of the circuit, since at any time

$$\dot{v}_c(t) = -\frac{1}{RC} v_c(t) - \frac{1}{C} i_L(t) + \frac{1}{RC} v_i(t)$$

$$\dot{i}_L(t) = \frac{1}{L} v_c(t)$$

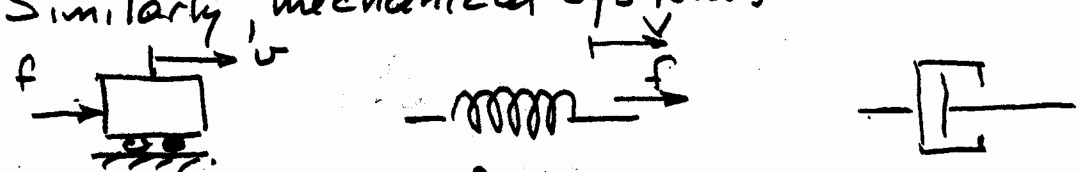
It doesn't matter how the system came to particular values of $v_c(t)$, $i_L(t)$.

- If we could save, then restore, the state, the system would be unaffected by the gap

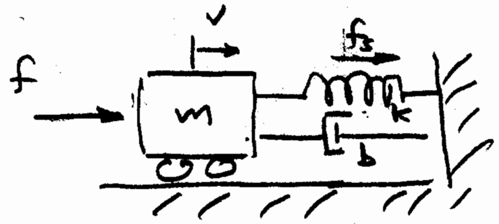


- A similar concept applies to multitasking on a computer. The state is the set of values in all CPU registers, stack pointers and data areas of a program. Save them, then restore at some future time, and the program simply carries on.

- Similarly, mechanical systems



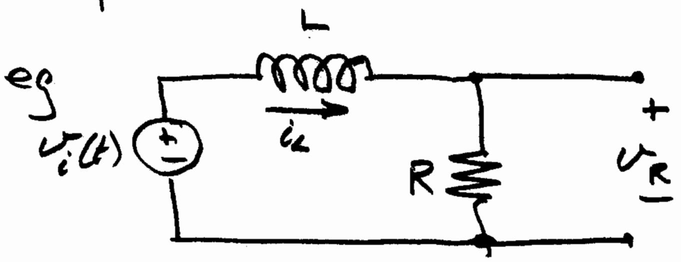
$\dot{v} = \frac{1}{m} f$ $\quad\quad$ $\dot{f} = k v$ $\quad\quad$ $f = b v$



use as state variables
- velocities of masses
- forces in springs

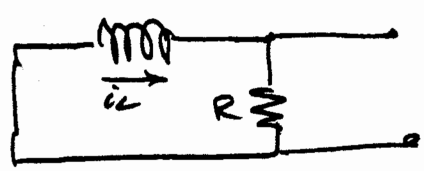
- If the initial state is non-zero, then the output will be non zero, even with no input. This is "zero input response."
- If initial state is zero (all cap voltages, inductor currents zero), system is at rest. Response to input is "zero state response"

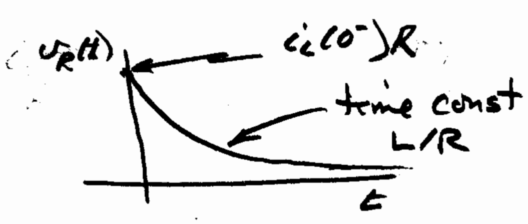
eg



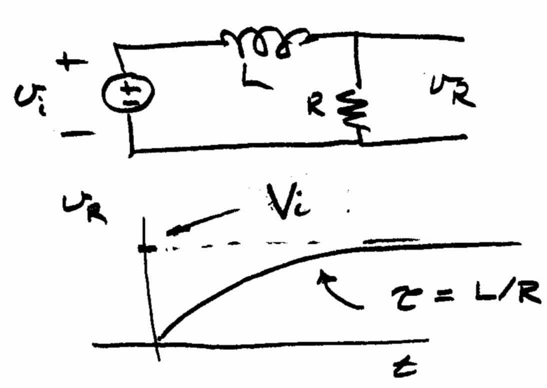
for step input $V_i u(t)$ and $i_L(0^-) \neq 0$

zero input response



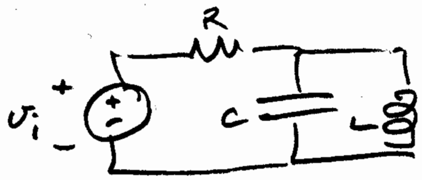


zero state response



- The state variables can be stacked in a state vector, leading to the state equations

example from p 1.3.5



$$\begin{bmatrix} v_c(t) \\ i_c(t) \end{bmatrix} = \underline{x}(t)$$

conventional notation $\underline{x}$

$$\begin{bmatrix} \dot{v}_c(t) \\ \dot{i}_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 1/RC \\ 0 \end{bmatrix} v_i(t)$$

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{b}\,u(t)$$

conventional notation

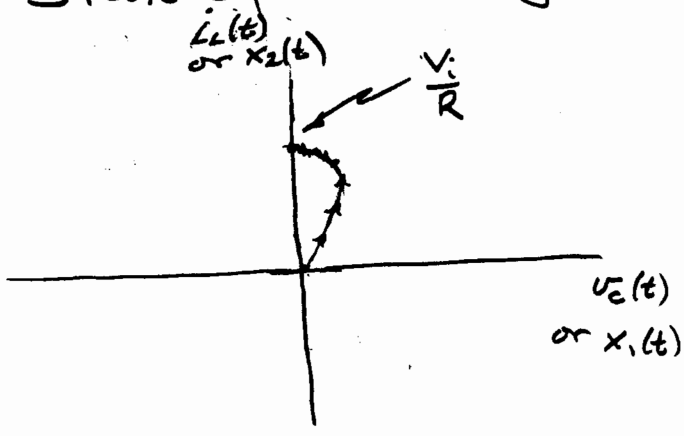
If output is i_R

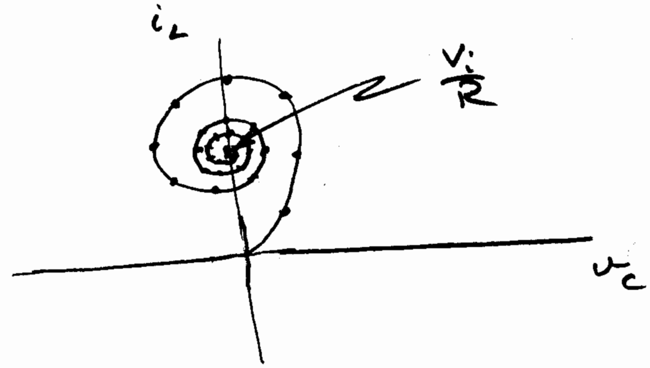
$$i_R(t) = \begin{bmatrix} -\frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 1/R \end{bmatrix} v_i(t)$$

$$y(t) = C\,\underline{x}(t) + d\,u(t)$$

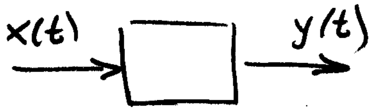
conventional

- State space trajectories in response to step





1.3.4 Order of Derivative on Right Hand Side

- Consider a system 

  described by $A[y(t)] = x(t)$

  Typical step responses

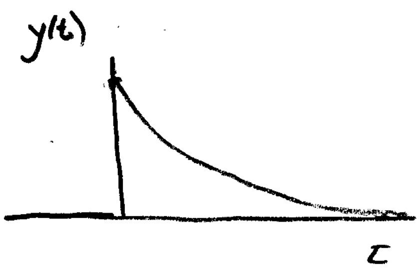
  1st order

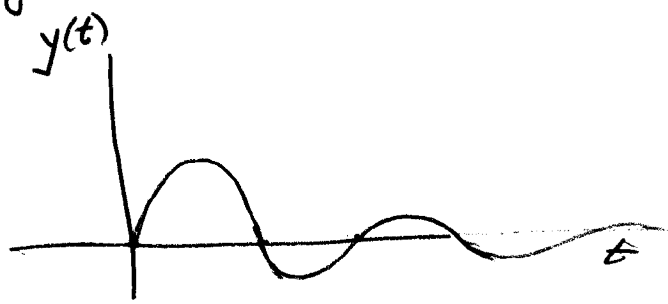
  2nd order

- If the RHS is replaced by $\dot{x}(t)$, the new system is described by

$$A[y(t)] = \dot{x}(t)$$

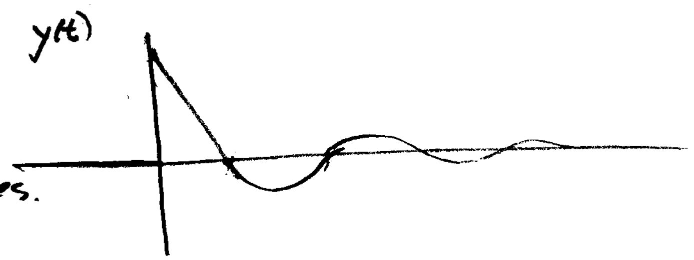
In form, this is just like the original system but driven by an input $\dot{x}(t)$. Hence response of new system is just the deriv. of response of the original





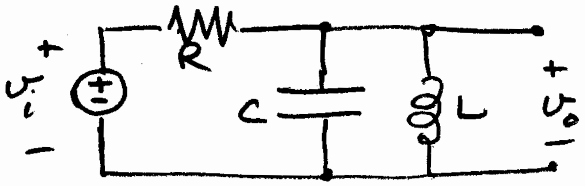
And for second order with $\ddot{x}(t)$ on RHS

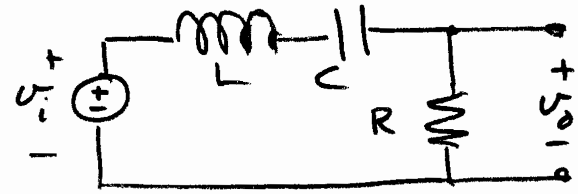
Order of discontinuity increases.



1.3.5 Standard Parameterization for Second Order

- Here are two second-order circuits





$$\ddot{v}_o + \frac{1}{RC}\dot{v}_o + \frac{1}{LC}v_o = \frac{1}{RC}\dot{v}_i \qquad \ddot{v}_o + \frac{R}{L}\dot{v}_o + \frac{1}{LC}v_o = \frac{R}{L}\dot{v}_i$$

The coefficients of the homogeneous DE depend on component values in a way determined by the structure of the circuit.

- To study behaviour of 2^{nd} order solutions in general, adopt this standard parameteriz'n

$$\ddot{v}_o + 2\zeta\omega_o\dot{v}_o + \omega_o^2 v_o = 0$$

ζ damping factor

ω_o natural frequency

The characteristic poly

$$s^2 + 2\zeta\omega_o s + \omega_o^2 = 0$$

has roots

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = -\zeta\omega_o \pm \frac{1}{2}\sqrt{(2\zeta\omega_o)^2 - 4\omega_o^2}$$

$$= \omega_o\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

The nature of the discriminant is set by ζ

- circuit 1

$\omega_0 = \frac{1}{\sqrt{LC}}$

$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$

decr w incr R

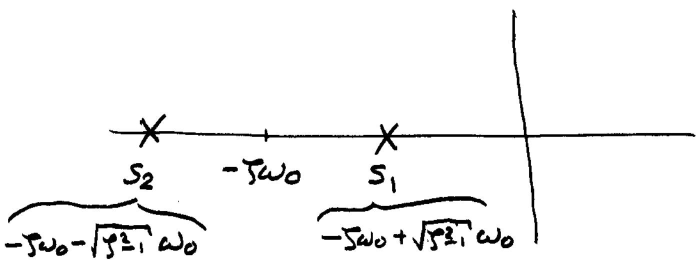
circuit 2

$\omega_0 = \frac{1}{\sqrt{LC}}$

$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$

incr w incr R

- Case $\zeta > 1$, overdamped, distinct real roots

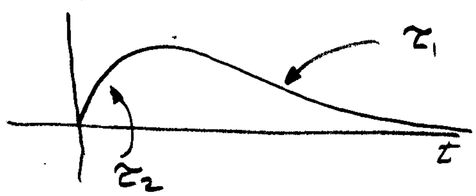


– Components of homog. sol'n.

$$e^{s_1 t} = e^{(-\zeta+\sqrt{\zeta^2-1})\omega_0 t} = e^{-t/\tau_1} \qquad \tau_1 = \frac{1}{\omega_0(\zeta - \sqrt{\zeta^2-1})}$$

$$e^{s_2 t} = e^{(-\zeta-\sqrt{\zeta^2-1})\omega_0 t} = e^{-t/\tau_2} \qquad \tau_2 = \frac{1}{\omega_0(\zeta + \sqrt{\zeta^2-1})}$$

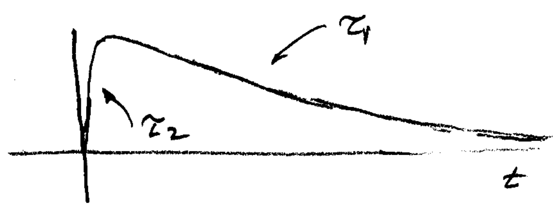
– Step response



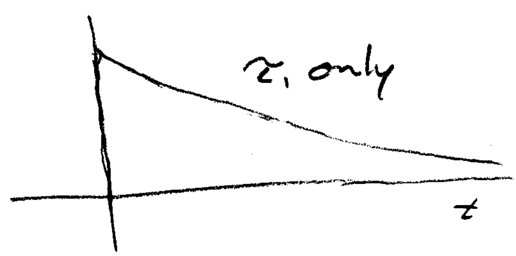
As ζ increases, τ_1 increases, asympt $\tau_1 \sim \frac{2\zeta}{\omega_0}$

τ_2 decreases, asympt $\tau_2 \sim \frac{1}{2\zeta\omega_0}$

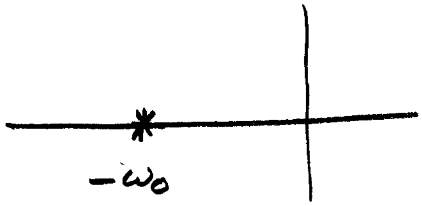
– Extreme case, large ζ



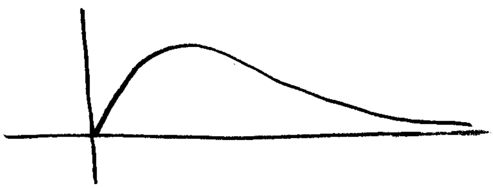
model as 1st order?



- Case $\zeta = 1$, critically damped, repeated root

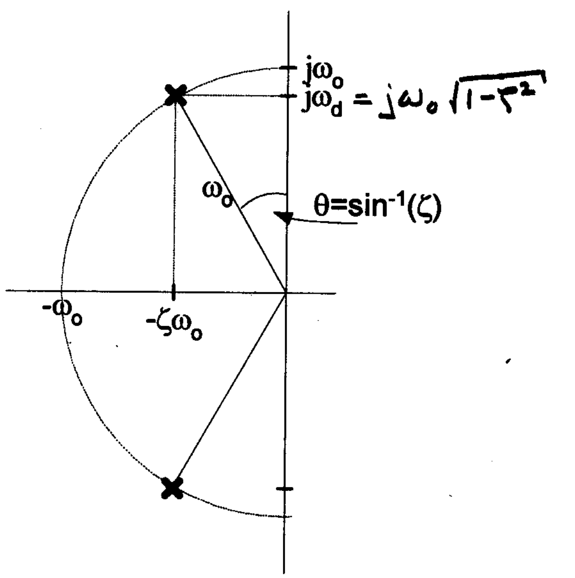


– step response $\quad t e^{s_1 t} = t e^{s_2 t} = t e^{-\omega_o t}$



- Case 3 $\quad \zeta < 1$, underdamped, complex roots

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \omega_o \left(-\zeta \pm j \sqrt{1-\zeta^2} \right)$$



ω_o natural freq, radius of the roots

ζ damping factor
$\zeta = \sin\theta$

ω_d damped natural freq, $\omega_d = \omega_o\sqrt{1-\zeta^2}$

– step response

$$G_1 e^{s_1 t} + G_2 e^{s_2 t}$$

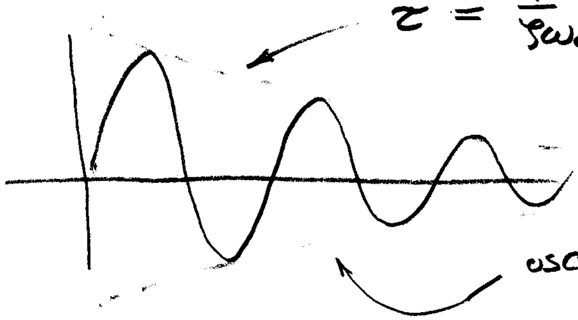
$$= e^{-\zeta\omega_0 t}\left(G_1 e^{j\omega_d t} + G_2 e^{-j\omega_d t}\right)$$

Note $G_2 = G_1^*$ to give real sum. For illustration, look at $G_1 = -j$ $G_2 = j$.

step response

$$e^{-\zeta\omega_0 t}\sin(\omega_d t) = e^{-\zeta\omega_0 t}\sin\left(\sqrt{1-\zeta^2}\,\omega_0 t\right)$$

$\tau = \frac{1}{\zeta\omega_0}$ light damping $\Rightarrow$ slow decay



oscillation freq $\omega_d \rightarrow \omega_0$ for light damping

- *In all three cases, we see from*

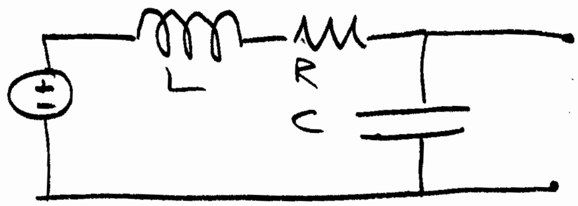
$$e^{s_i t} = e^{(-\zeta \pm \sqrt{\zeta^2 - 1})\omega_0 t}$$

that – ζ sets the "shape" of the response

– ω_0 sets the time scale

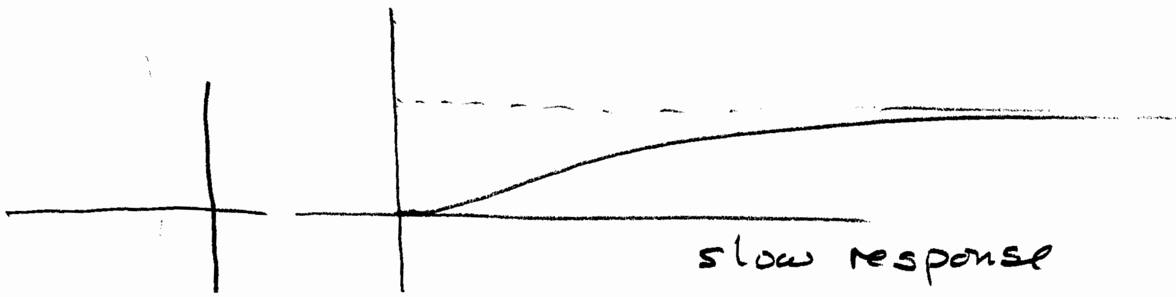
1.3.14

• For



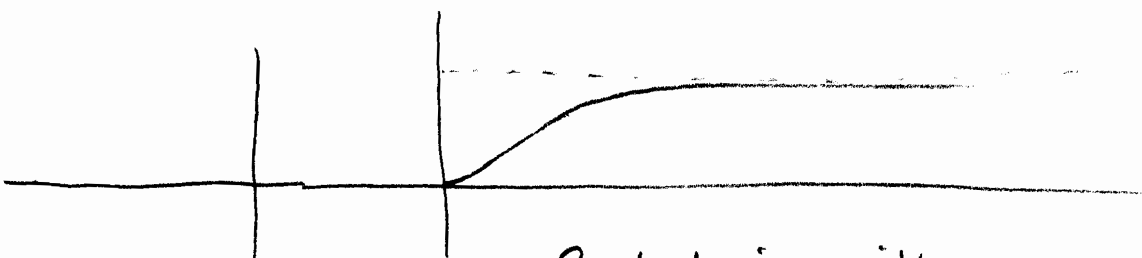
or any other 2^{nd} order circuit

the step response clearly shows the effect of damping factor



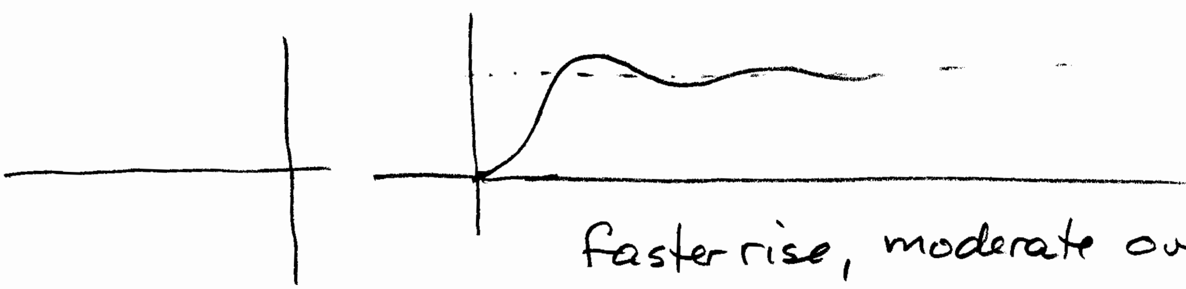
$\zeta >> 1$

slow response



$\zeta = 1$

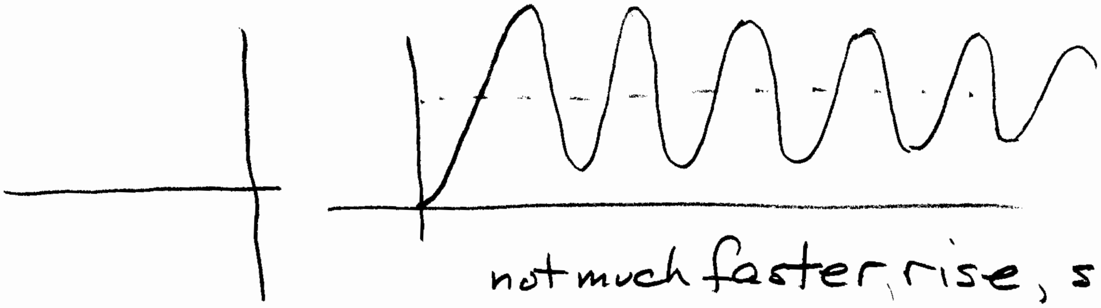
fastest rise with no overshoot



$\zeta < 1$

$\zeta \approx 1/\sqrt{2}$

faster rise, moderate overshoot



$\zeta << 1$

not much faster rise, slow convergence