1.4 The Convolution Input-Output Relation

DeCarlo + Lin Chapt 16

- For systems describable by a LDECC (linear DE with constant coeffs), we can determine the response to any input by solving the DE.

- Another way: if we already have the impulse response $h(t)$, then we can use it to calculate the response to any input without solving the DE. Even better, it works for any LTI system, not just ones with LDECC dynamics.

$\longrightarrow$ the convolution integral

operation: convolution

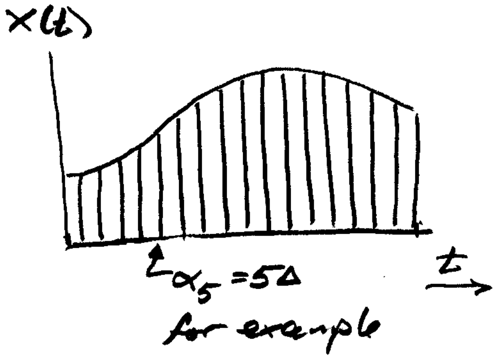
verb infinitive: to convolve.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ?$$

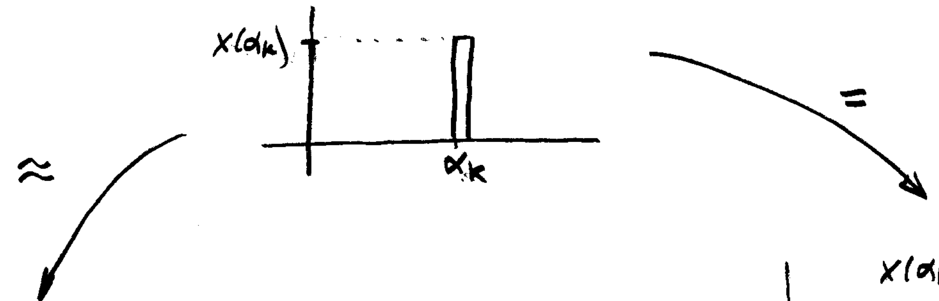
in terms of $x(t)$, $h(t)$

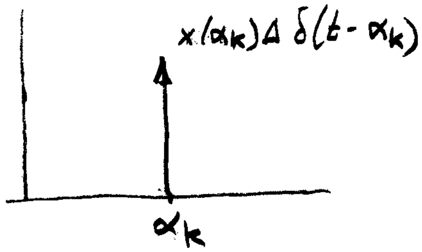
- Approx $x(t)$ as staircase, strips Δ wide:

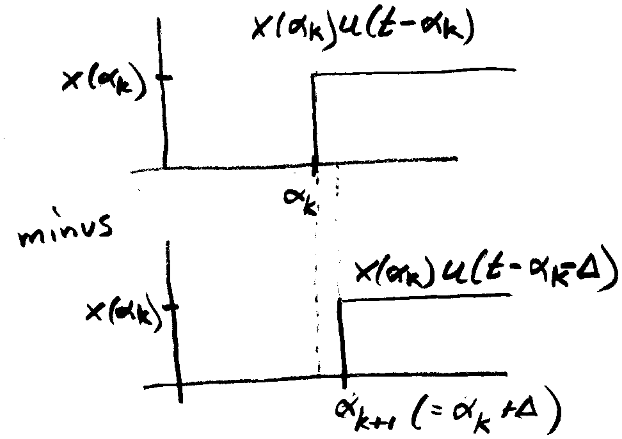
  $\alpha_k = k\Delta$, strip area $\approx x(\alpha_k)\Delta$



- Use superposition, get response to input strip k alone. Two alternative routes to the same result.







which produces at time t

$x(\alpha_k)\Delta\, h(t-\alpha_k)$

which produces at time t

$x(\alpha_k)\big(g(t-\alpha_k)-g(t-\alpha_k-\Delta)\big) \approx x(\alpha_k)\, h(t-\alpha_k)\Delta$

by superposition: $$y(t) = \sum_{k=0}^{t/\Delta} x(\alpha_k)\, h(t-\alpha_k)\,\Delta$$

excitation time α

limit as $\Delta \to 0$: $$y(t) = \int_0^t x(\alpha)\, h(t-\alpha)\, d\alpha$$

observation time t

- A change of variable gives an alternative interpretation:

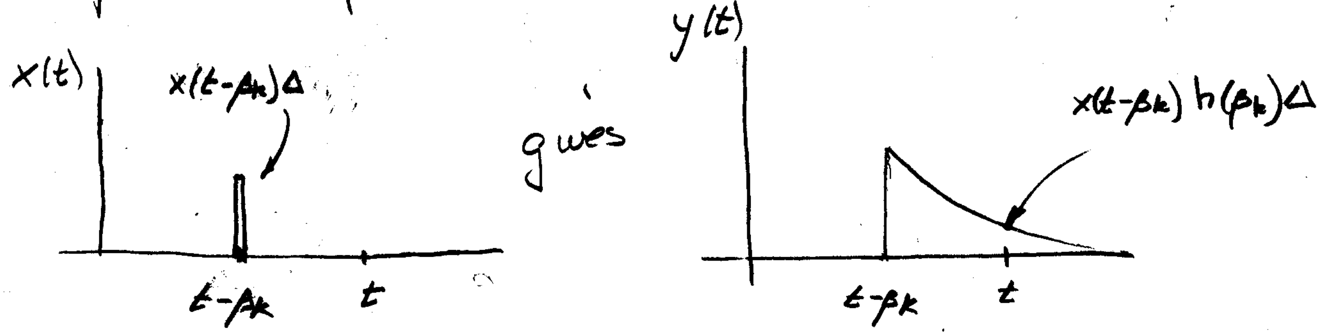
delay time

$$y(t) = \int_0^t x(\alpha)\, h(t-\alpha)\, d\alpha \underset{\alpha = t-\beta}{=} \int_0^t x(t-\beta)\, h(\beta)\, d\beta$$

observation time

$\boxed{\text{Note } d\beta = -d\alpha \text{ but integration limits reversed, too}}$

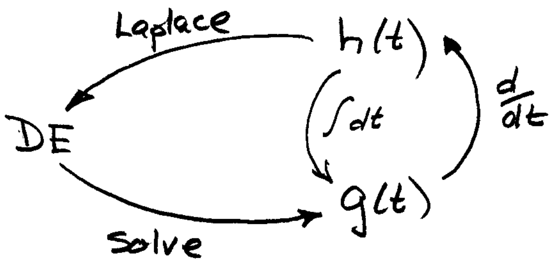
Pictorially, for a decaying exponential impulse response:



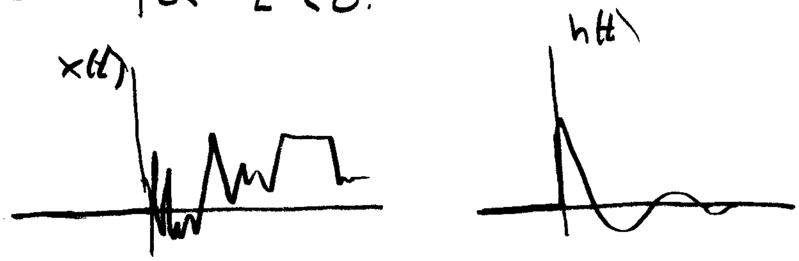
- Significance of convolution relation:
  - If you know the impulse response, you can obtain the response to <u>any</u> input from it. (zsr, that is)
  - Same with step response, since

$$\begin{tikzcd} h(t) \arrow[r, bend left, "\int dt"] & g(t) \arrow[l, bend left, "d/dt"] \end{tikzcd}$$

– If you can obtain response from either DE or impulse response, then you should be able to work back to DE from $h(t)$.



- Integration limits. Since circuit is causal, $h(t) \equiv 0$ for $t < 0$. Assume input is also zero for $t < 0$.



$$y(t) = \int_0^t x(t-\beta)\, h(\beta)\, d\beta$$

$$= \int_0^t x(\alpha)\, h(t-\alpha)\, d\alpha$$

Or use $\int_{-\infty}^{\infty}$ if you use $h(t) \equiv 0,\ t<0$

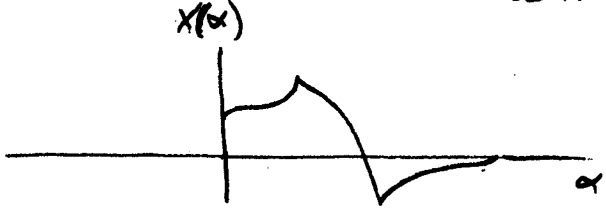
$x(t) \equiv 0,\ t<0$

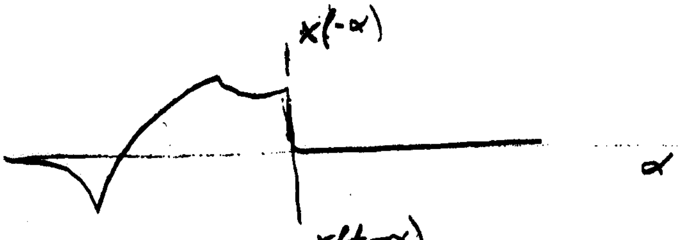
in the integrand

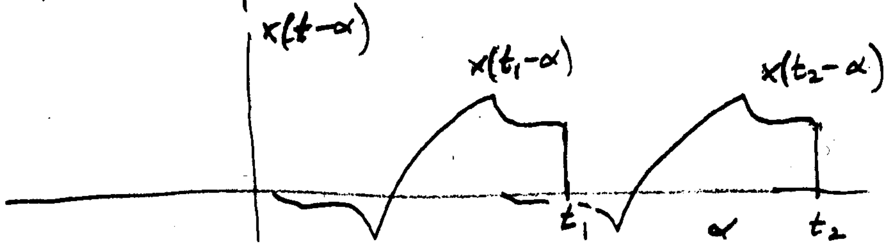
"Flip, slide and integrate"

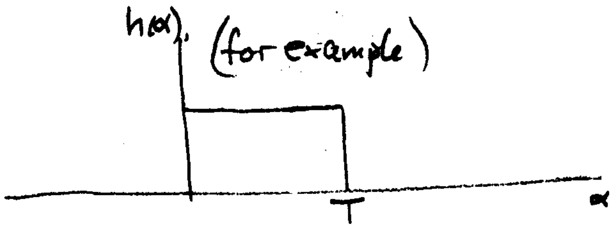
$$y(t) = \int_0^t h(\alpha)\, \underbrace{x(t-\alpha)}\, d\alpha$$

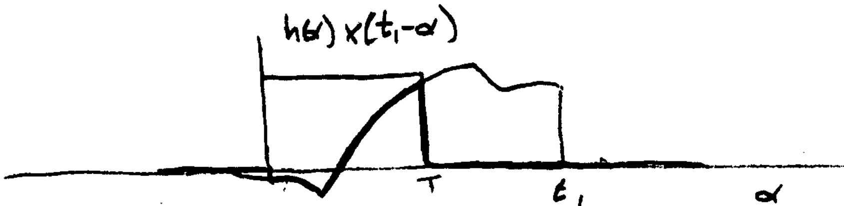
sketch this as a function of α, as it appears in the integrand

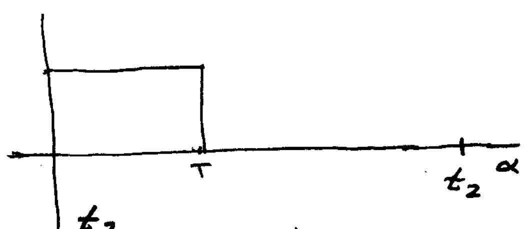




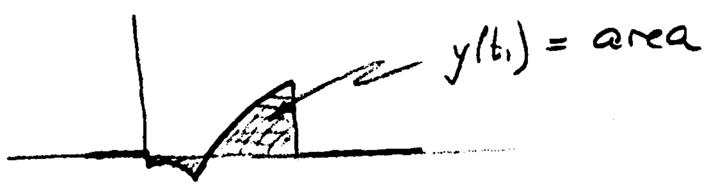


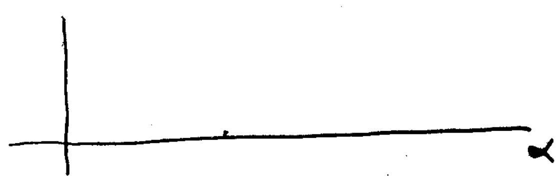


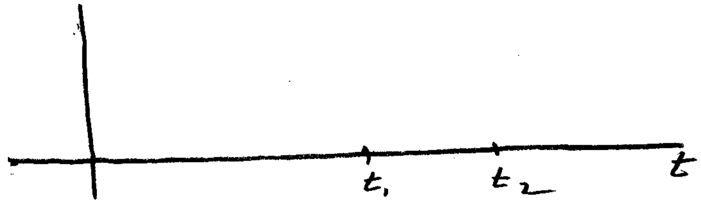




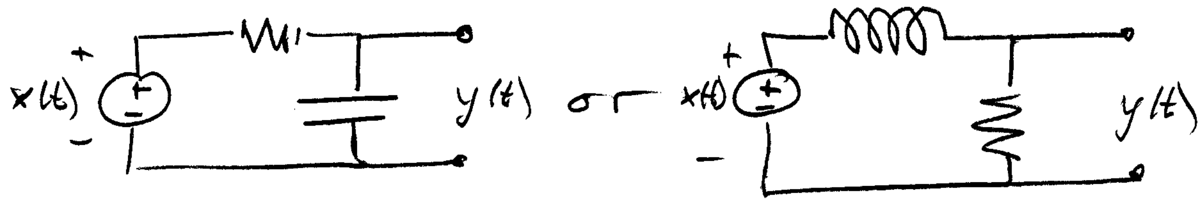
$$y(t_1) = \int_0^{t_1} h(\alpha)\, x(t_1-\alpha)\, d\alpha \qquad y(t_2) = \int_0^{t_2} h(\alpha)\, x(t_1-\alpha)\, d\alpha$$







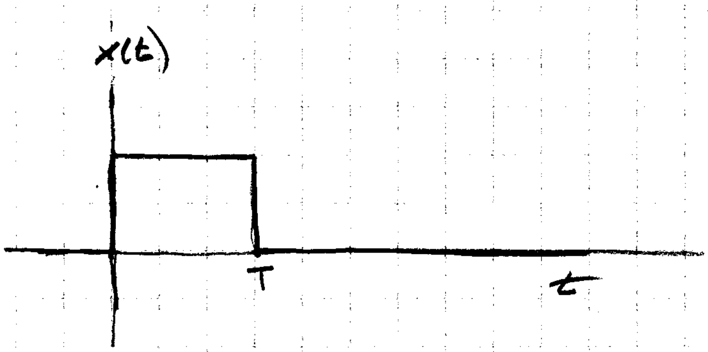
- Try it with an easy one: the response of a first order lowpass, like

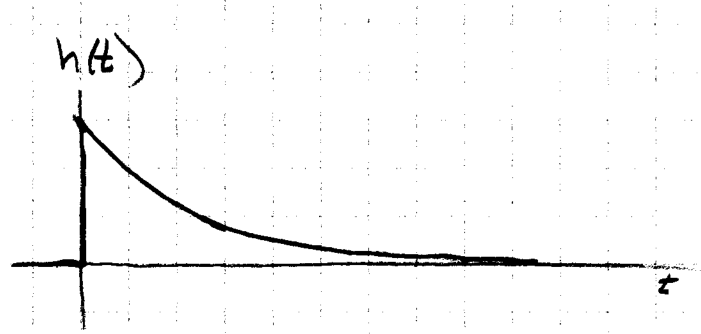


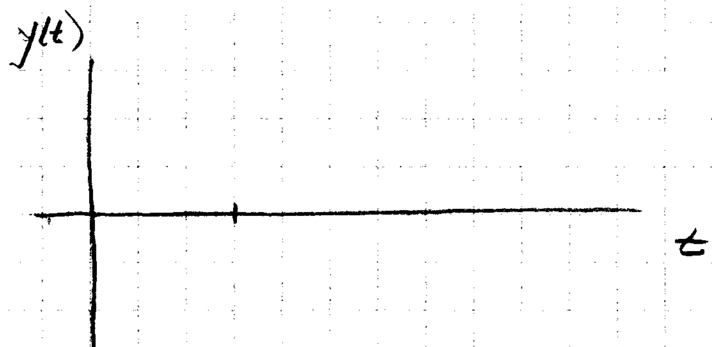
We'll flip and slide the input, so

$$y(t) = \int_0^\infty x(t-\beta)\, h(\beta)\, d\beta$$

but it doesn't matter which one.





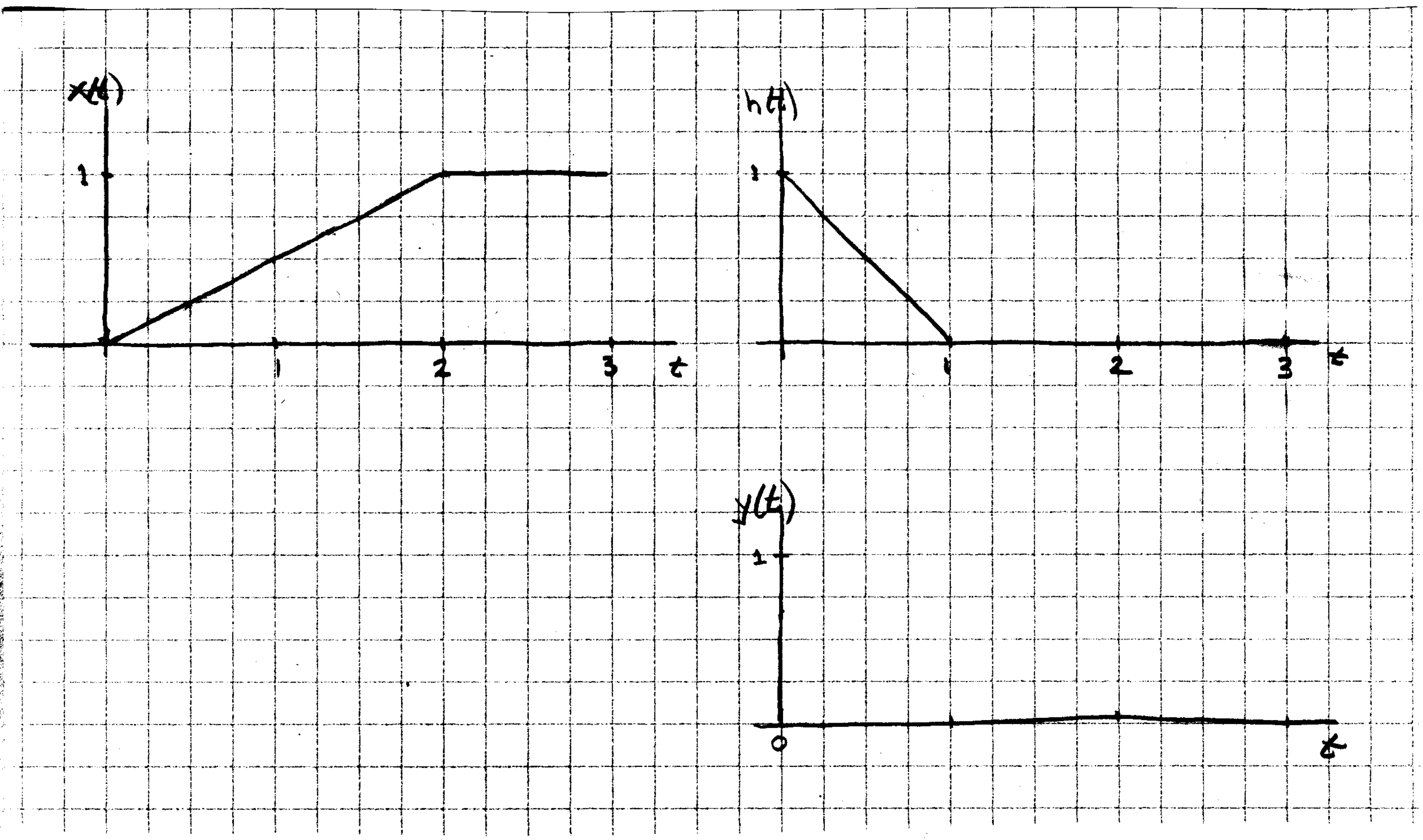


Units of $h(t)$?

- The output at some time t is the area of the <u>product</u> of

$$x(t-\alpha)\,h(\alpha) \quad \text{or} \quad x(\alpha)\,h(t-\alpha)$$

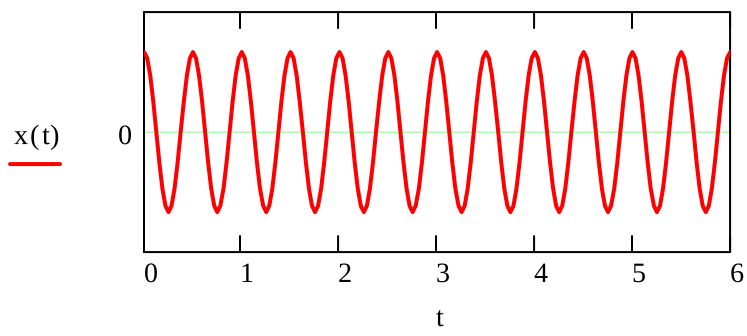
$$y(t) = \int_0^t x(t-\alpha)\,h(\alpha)\,d\alpha = \int_0^t x(\beta)\,h(t-\beta)\,d\beta$$

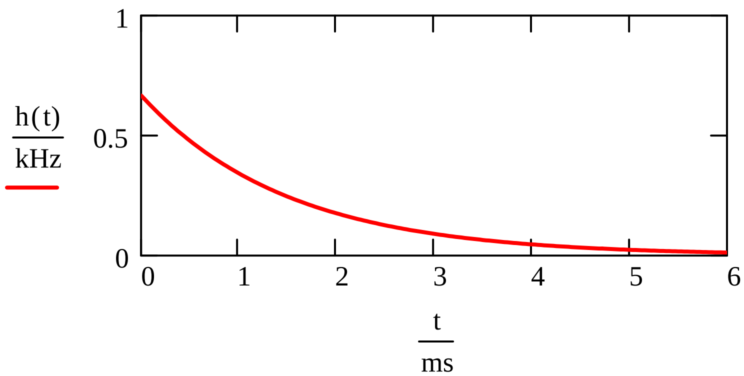


Example: First order lowpass system (e.g., an RC circuit), input is a cosine switched on at time zero. Find the response by convolution - "flip, slide and integrate" - input x(t) with impulse response h(t).

$f_c := 2\cdot\text{kHz}$ $\quad x(t) := \cos(2\cdot\pi\cdot f_c\cdot t)$ $\quad \tau := 1.5\cdot\text{ms}$ $\quad h(t) := \frac{1}{\tau}\cdot e^{-\frac{t}{\tau}}$

$t := 0\cdot\text{ms}, \frac{1}{16\cdot f_c} .. 6\cdot\text{ms}$



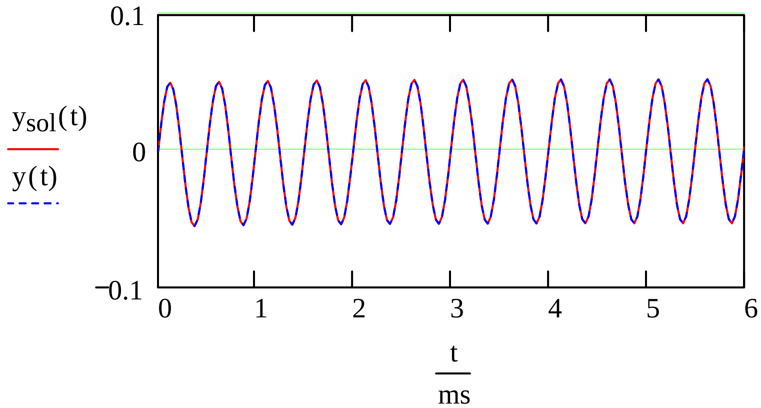


$$y(t) := \int_{0\cdot\text{ms}}^{t} x(\alpha)\cdot h(t-\alpha)\,d\alpha$$

by convolution

$$y_{sol}(t) := \frac{1}{1+\tau^2\cdot(2\cdot\pi\cdot f_c)^2}\cdot\left(\cos(2\cdot\pi\cdot f_c\cdot t) + 2\cdot\pi\cdot f_c\cdot\tau\cdot\sin(2\cdot\pi\cdot f_c\cdot t) - \exp\left(-\frac{t}{\tau}\right)\right)$$

solution of differential equation

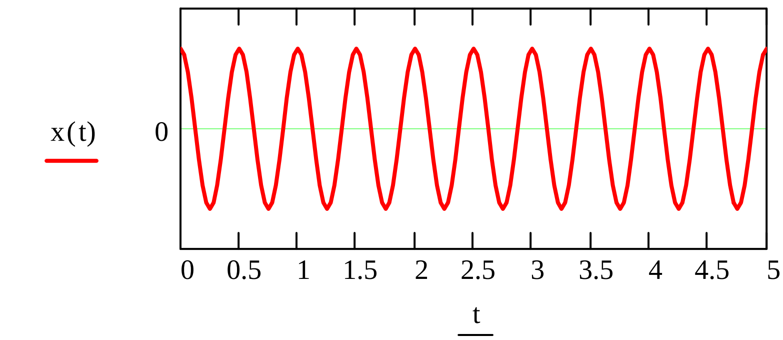


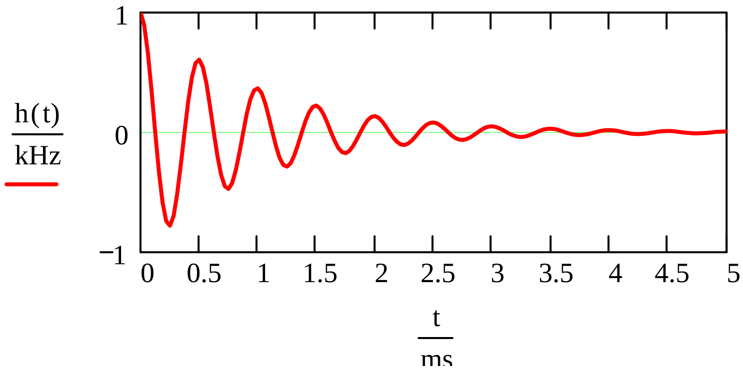
See - the convolution and the DE solution are identical (the traces fall on top of each other).

Example: sine wave input to a filter tuned to the same frequency (i.e., resonance)

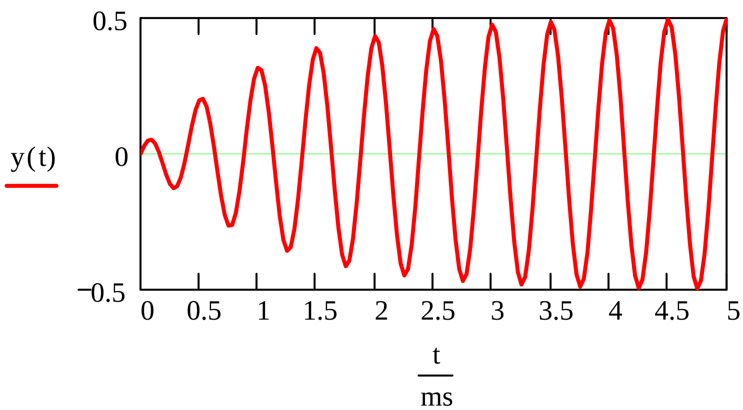
$f_c := 2 \cdot kHz$ $\quad x(t) := \cos(2 \cdot \pi \cdot f_c \cdot t)$ $\quad \tau := 1 \cdot ms$ $\quad h(t) := \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}} \cdot \cos(2 \cdot \pi \cdot f_c \cdot t)$

$t := 0 \cdot ms, \frac{1}{16 \cdot f_c} .. 5 \cdot \tau$





$$y(t) := \int_{0 \cdot ms}^{t} x(\alpha) \cdot h(t - \alpha) \, d\alpha$$

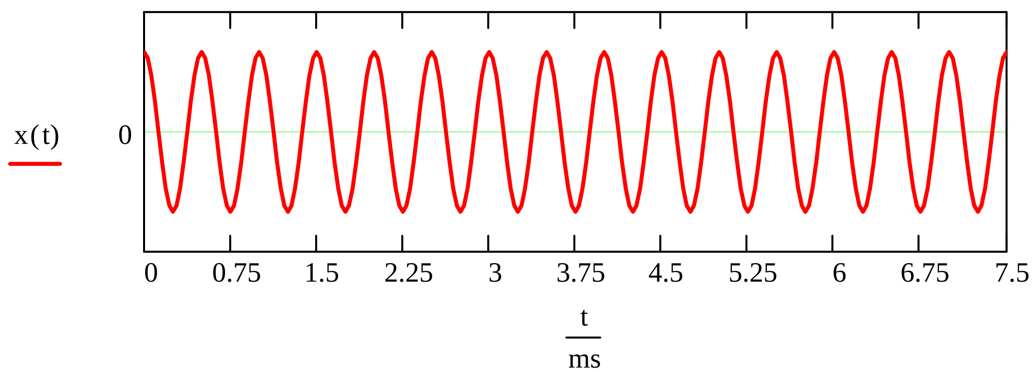


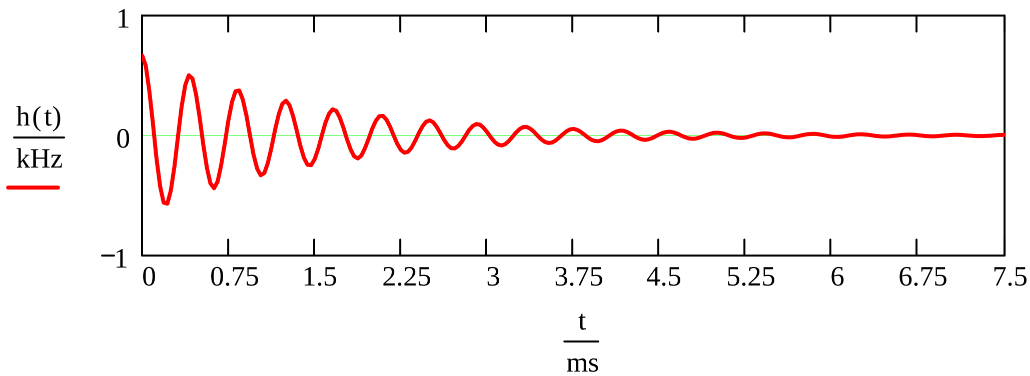
<u>Example:</u> sine wave input to a filter tuned to a slightly different frequency (produces beats)

$f_c := 2\cdot kHz$ $\quad x(t) := \cos(2\cdot\pi\cdot f_c\cdot t)$

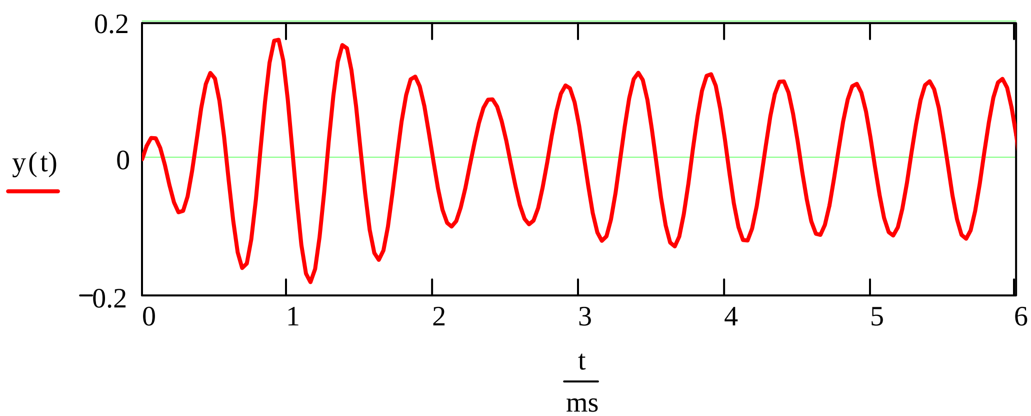
$f_o := 1.2\cdot f_c$ $\quad \tau := 1.5\cdot ms$ $\quad h(t) := \frac{1}{\tau}\cdot e^{-\frac{t}{\tau}}\cdot\cos(2\cdot\pi\cdot f_o\cdot t)$

$t := 0\cdot ms, \frac{1}{16\cdot f_c} .. 5\cdot\tau$





$$y(t) := \int_{0\cdot ms}^{t} x(\alpha)\cdot h(t-\alpha)\, d\alpha$$



- Properties of convolution.

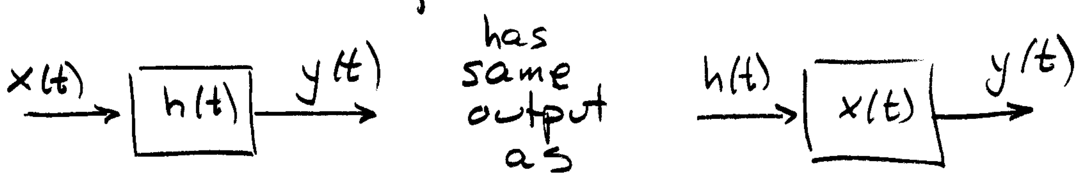
Denote $y(t) = \int x(\alpha)\, h(t-\alpha)\, d\alpha$ as $y(t) = x(t) \circledast h(t)$

– It's commutative:

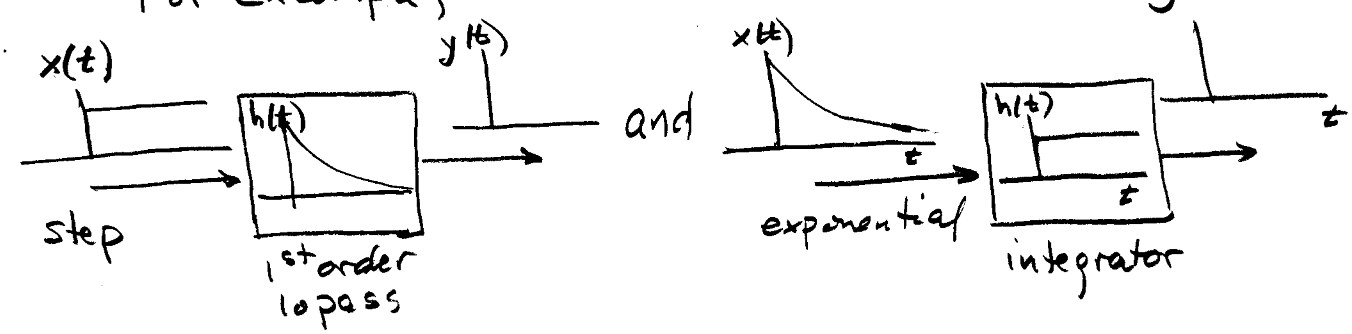
$$y(t) = \int x(\alpha)\, h(t-\alpha)\, d\alpha = \int h(\beta)\, x(t-\beta)\, d\beta$$

so $y(t) = x(t) \circledast h(t) = h(t) \circledast x(t)$

This implies that input and impulse response can be exchanged.

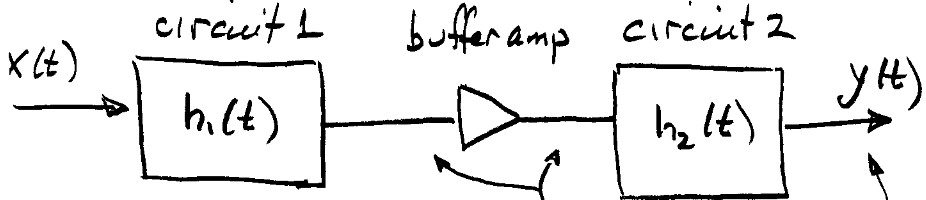


For example,



1.4.10

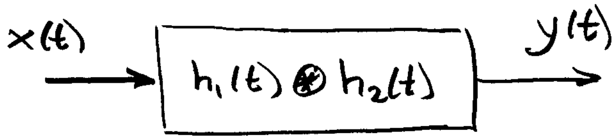
– Cascade connection (no loading)



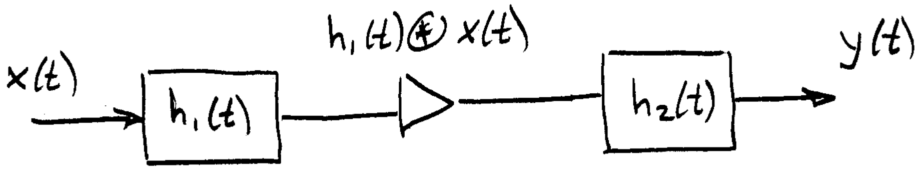
If $x(t) = \delta(t)$, then this is $h_1(t)$ and $y(t) = h_1(t) \circledast h_2(t)$

Therefore the combined circuit is equivalent to

$$h(t) = h_1(t) \circledast h_2(t)$$



– It's associative



$$y(t) = h_2(t) \circledast \left(h_1(t) \circledast x(t)\right)$$

$$= \left(h_2(t) \circledast h_1(t)\right) \circledast x(t) \qquad \text{cascade}$$

1.4.11

- Combined commutative and associative properties give these all the same response (assume buffer amps)

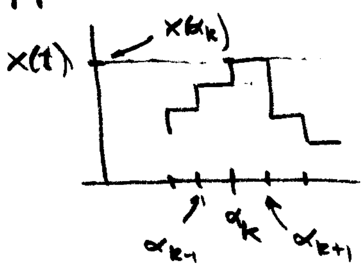
$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

$$x(t) \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow y(t)$$

$$h_2(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

- Derivative property

Approximate the input by steps, instead of strips.



At time α_k, its a step

$$\left(x(\alpha_k) - x(\alpha_{k-1})\right) u(t - \alpha_k)$$

Step k produces output $\left(x(\alpha_k) - x(\alpha_{k-1})\right) g(t - \alpha_k)$

Total:

$$y(t) = \sum_{k=0}^{t/\Delta - 1} \left(x(\alpha_k) - x(\alpha_{k-1})\right) g(t - \alpha_k) = \sum_{k=0}^{t/\Delta - 1} \frac{x(\alpha_k) - x(\alpha_{k-1})}{\Delta} g(t - \alpha_k)\, \Delta$$

Limit as $\Delta \to 0$:

$$y(t) = \int_0^t \dot{x}(\alpha) g(t - \alpha)\, d\alpha$$

1.4.12

Using notation $f^{(1)}(t) = \frac{d}{dt} f(t)$

$$f^{(-1)}(t) = \int_{-\infty}^{t} f(\alpha)\,d\alpha$$

we have

$$y(t) = \int x(\alpha)\, h(t-\alpha)\,d\alpha = \int x^{(1)}(\alpha)\, g(t-\alpha)\,d\alpha$$

$$= \int x(t-\beta)\, h(\beta)\,d\beta = \int x^{(-1)}(t-\beta)\, h^{(1)}(\beta)\,d\beta$$

Also obtainable using integration by parts

- Convolution with impulses:

$$x(t) \circledast \delta(t) = \int x(\alpha)\,\delta(t-\alpha)\,d\alpha = x(t)$$

$$x(t) \circledast \delta(t-\tau) = \int x(\alpha)\,\delta(t-\alpha-\tau)\,d\alpha = x(t-\tau)$$

$$x(t) \longrightarrow \boxed{h(t)=\delta(t)} \longrightarrow x(t) \qquad\qquad x(t) \longrightarrow \boxed{h(t)=\delta(t-\tau)} \longrightarrow x(t-\tau)$$

unity "filter"

- Read all of DC & L Chapter 16 (except 16.3) and work through the examples.