2. LAPLACE TRANSFORMS

- The Laplace Transform converts a function of time $x(t)$ to a counterpart function of frequency $X(s)$. The Inverse Laplace Transform takes you back again, from $X(s)$ to $x(t)$.

- Why do we care?
  - The LT dramatically simplifies analysis <u>and design</u> of LTI circuits
  - It provides insights into the frequency makeup of signals and frequency response of circuits, as well as behaviour of feedback systems.

- You will learn to
  - calculate the LT — calculate the ILT
  - analyse circuit behaviour with LT
  - calculate frequency response with LT

and you will see how time and frequency descriptions are linked.

- prepares you for filter design, control systems etc

2.1 Plausibility of Integral Transforms

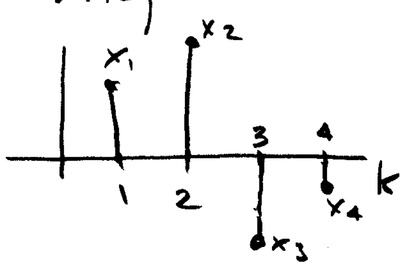
- The Laplace transform can be understood readily in the context of Fourier transforms. When you learn the FT, read <u>The Laplace Transform</u> in Supplementary Notes and Demos.

In the meantime, an analogy will make the LT seem less arbitrary.

- Vectors and functions are similar:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

indexed by an integer variable



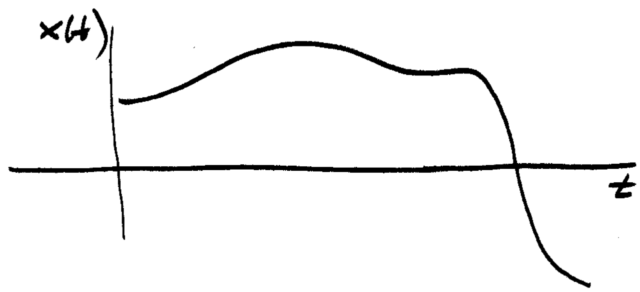
x_k

or $x(k)$, if you like

for every $k \in [1..N]$ there's an x_k

$x(t)$

"indexed" by a real variable



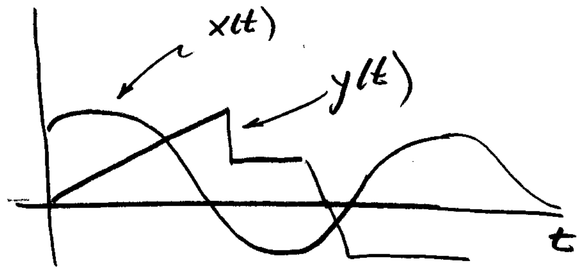
for every t in $[0, \infty)$ there's an $x(t)$

- Summation is the counterpart of integration

$$\sum_{k=1}^{N} x_k \qquad\qquad \int_0^{\infty} x(t)\,dt$$

- They both have inner (dot) products that are the summation/integration of the point-by-point products of two vectors/functions

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$



$$(\underline{x}, \underline{y}) = \sum_{k=1}^{N} x_k y_k \qquad\qquad (x(t), y(t)) = \int_0^{\infty} x(t)\,y(t)\,dt$$

to give a scalar result.

- The similarities are profound and very ... very ... useful.

- You are used to transforming vectors with a change of basis, using a matrix multiplication

$$\underline{\tilde{x}} = W\,\underline{x}$$

$$i\downarrow\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \vdots\\ \tilde{x}_N\end{bmatrix} = \begin{array}{c} \xrightarrow{\;k\;} \\ i\downarrow\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N}\\ w_{21} & w_{22} & \cdots & w_{2N}\\ \vdots & \vdots & & \vdots\\ w_{N1} & w_{N2} & \cdots & w_{NN}\end{bmatrix}\end{array}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_N\end{bmatrix}k\downarrow$$

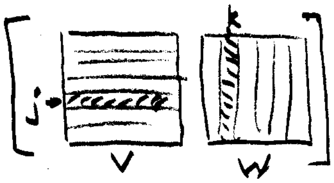
$$\tilde{x}_i = \sum_{k=1}^{N} w_{ik}\,x_k = \left(\text{row } i \text{ of } W, \underline{x}\right)$$

- W is indexed by (is a function of) two variables, i and k.

- If W is non-singular (i.e. full rank, independent rows, indep cols, no zero eigenval, etc) then it has an inverse, and you can get back to $\underline{x}$ from $\underline{\tilde{x}}$. If $V = W^{-1}$, then

$$\underline{\tilde{\tilde{x}}} = V\underline{\tilde{x}} = VW\underline{x} = I\underline{x} = \underline{x} \qquad \left(I_{ik} = \delta_{ik}\right)$$

$$[VW]_{jk} = $$



$$= \sum_i v_{ji}\,w_{ik} = \delta_{jk}$$

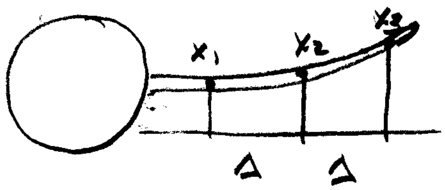
or by detailed summations

$$\tilde{x}_j = \sum_i v_{ji}\tilde{x}_i = \sum_i v_{ji}\left(\sum_k w_{ik}x_k\right) = \sum_i\sum_k v_{ji}w_{ik}x_k$$

$$= \sum_k\left(\sum_i v_{ji}w_{ik}\right)x_k = \sum_k \delta_{jk}x_k = x_j$$

Note summation indices are like "dummy variables"

• Example – flexing of an aircraft wing



displacements

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For analytical purposes, you may be more interested in

$\tilde{x}_1 = \frac{1}{3}(x_1 + x_2 + x_3)$ mean displacement

$\tilde{x}_2 = \frac{1}{2\Delta}(x_3 - x_1)$ est. derivative at centre

$\tilde{x}_3 = \frac{1}{\Delta^2}(x_3 - 2x_2 + x_1)$ est 2nd derivat centre

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2\Delta & 0 & \frac{1}{2\Delta} \\ \frac{1}{\Delta^2} & -\frac{2}{\Delta^2} & \frac{1}{\Delta^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

indexed by i $\longrightarrow$ $\underset{\sim}{\tilde{x}} = \underset{\text{indexed by } i,k}{W} \cdot \underline{x}$ $\longleftarrow$ indexed by k

Its reversible. You can always recover

$$\underline{x} = W^{-1}\underset{\sim}{\tilde{x}}$$

- What is the counterpart for functions?
  - Use a kernel that depends on two variables $w(s,t)$
  - Transform $x(t)$ by

$$\tilde{x}(s) = \int w(s,t)\, x(t)\, dt$$

For any s value, $\tilde{x}(s)$ is the inner prod in t of $w(s,t)$ and $x(t)$.

  - Why do it? Perhaps $\tilde{x}(s)$ is more convenient.
  - Can we get back again?

Depends on whether $w(s,t)$ has an inverse $v(r,s)$, another integral transform.

$$\tilde{\tilde{x}}(r) = \int_s v(r,s)\, \tilde{x}(s)\, ds = \int_s \int_t v(r,s)\, w(s,t)\, x(t)\, dt\, ds$$

$$= \int_t \left(\int_s v(r,s)\, w(s,t)\, ds \right) x(t)\, dt$$

If $\int_s v(r,s)\, w(s,t)\, ds = \delta(r-t)$

then $\tilde{\tilde{x}}(r) = \int_t \delta(r-t)\, x(t)\, dt = x(r)$

- So functions can be transformed by a "change of basis". If there is such a thing as an inverse, it's another integral transform.

$$
\begin{tikzcd}
x(t) \arrow[rr, bend left, "\int w(s,t)\, x(t)\, dt"] & & \tilde{x}(s) \arrow[ll, bend left, "\int v(t,s)\, \tilde{x}(s)\, ds"]
\end{tikzcd}
$$