2.2 The Laplace Transform

$\boxed{\text{DC\&L Chapt 13}}$

- The LT is an invertible integral transform. You have used some of its properties before without realising it.

- And here's why we like it:

  - Time domain

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) \circledast h(t)$$

Laplace transform domain

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

output is just $Y(s) = H(s)\,X(s)$

where

$$X(s) = \mathcal{L}[x(t)]$$

$$H(s) = \mathcal{L}[h(t)]$$

$$Y(s) = \mathcal{L}[y(t)]$$

so if we have $H(s)$, we find circuit response to an input $x(t)$ like this

$$x(t) \xrightarrow{\mathcal{L}} X(s) \xrightarrow{H(s)X(s)} Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t)$$

- Another nice property:

If $\mathcal{L}[x(t)] = X(s)$

then $\mathcal{L}[\dot{x}(t)] = s\,X(s)$ $\quad$ (if $x(0^-) = 0$)

example

$$\dot{y}(t) + 2y(t) = 3\,\dot{x}(t)$$

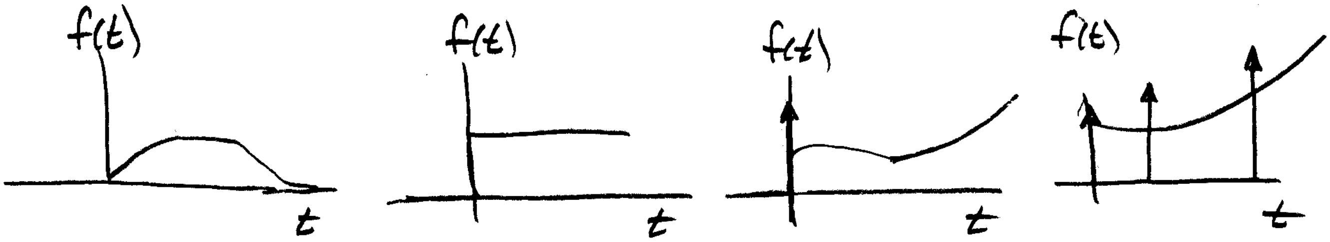
$$\mathcal{L}: \quad s\,Y(s) + 2\,Y(s) = 3s\,X(s)$$

or

$$Y(s) = \frac{3s}{s+2}\,X(s)$$

so it simplifies the solution of DEs.

- We'll work only with one sided functions



so it's the "one sided Laplace transform".

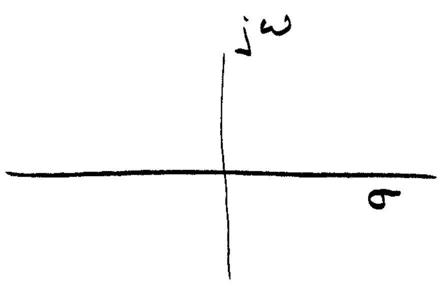
And here it is

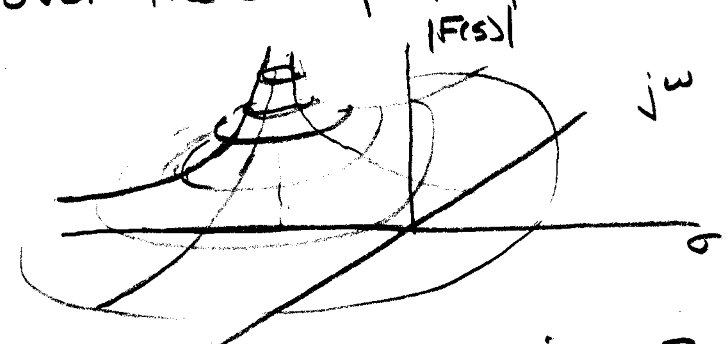
$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

where the transform variable may be complex

$$s = \sigma + j\omega$$

so $F(s)$ is defined over the complex plane



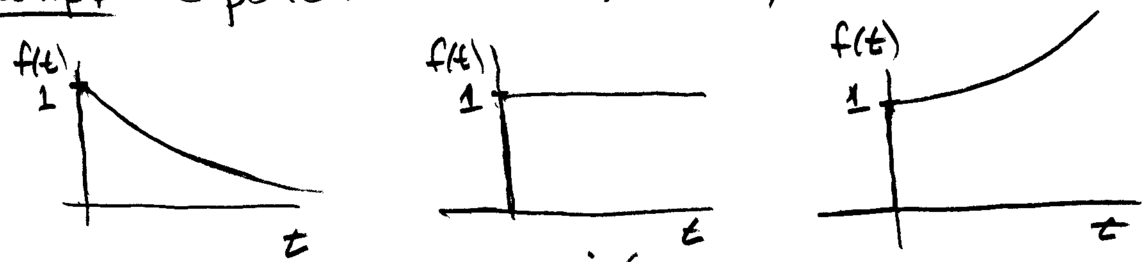


magnitude of a simple $F(s)$

There may be some constraints on values of s for convergence of the integral.

- Laplace transforms of some elementary functions

<u>example</u> exponential $f(t) = e^{at}$, $t \geq 0$



$a < 0$

special case
$a = 0$
unit step $u(t)$

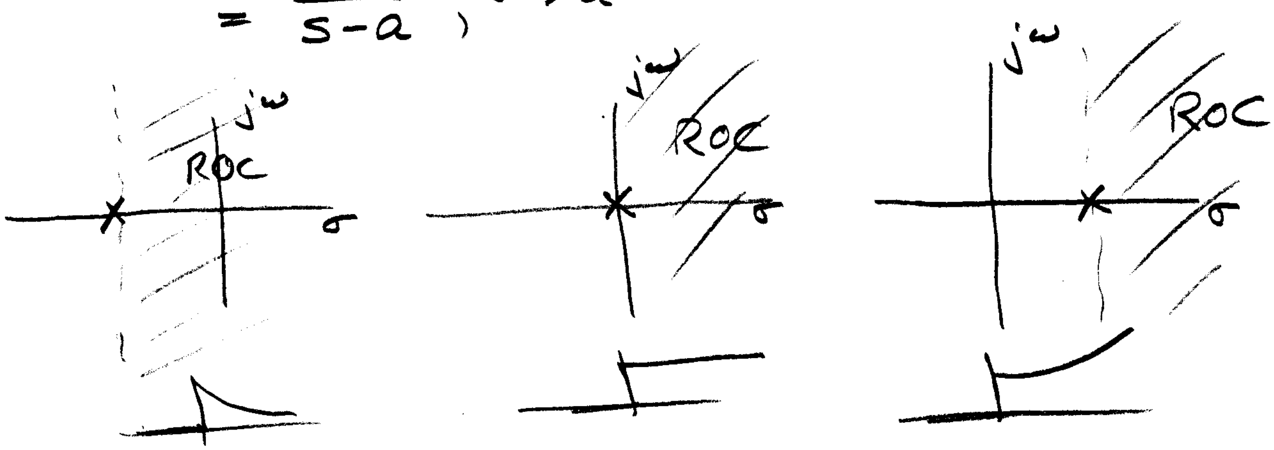
$$F(s) = \int_{0^-}^{\infty} e^{at} e^{-st} dt = \int_{0^-}^{\infty} e^{(a-\sigma)t} e^{-j\omega t} dt$$

For convergence, need $\sigma > a$ (region of convergence)

For $\sigma > a$,

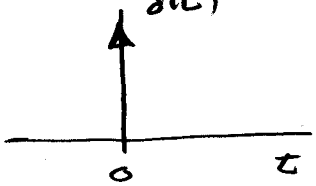
$$F(s) = \int_{0^-}^{\infty} e^{(a-s)t} dt = \left(\frac{e^{(a-s)t}}{a-s} \right|_{0^-}^{\infty} = 0 - \frac{1}{a-s}$$

$$= \frac{1}{s-a}, \quad \sigma > a$$



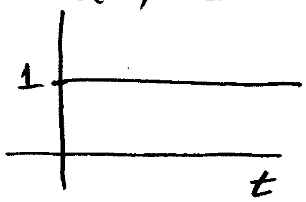
example monomials – unit impulse, unit step, unit ramp, ...

$\delta(t)$



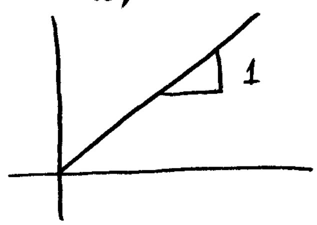
$$\int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-s0} = 1$$

$u(t) = 1$



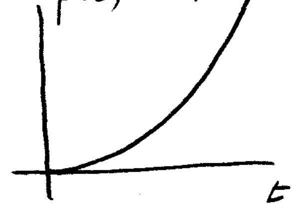
$$\int_{0^-}^{\infty} u(t)\, e^{-st}\, dt = \frac{1}{s}, \quad \sigma > 0$$

$r(t) = t$



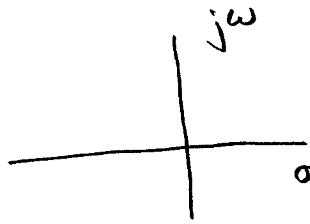
$$\int_{0^-}^{\infty} r(t)\, e^{-st}\, dt = \int_{0^-}^{\infty} t\, e^{-st}\, dt = \left(-t\frac{e^{-st}}{s}\right\Big|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} e^{-st}\, dt = \frac{1}{s^2}, \quad \sigma > 0$$

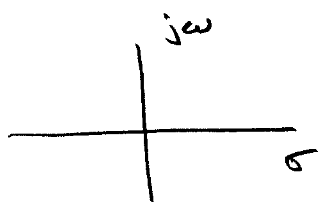
$p(t) = t^2/2$

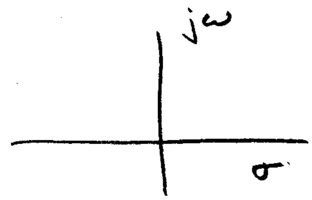


$$\int_{0^-}^{\infty} p(t)\, e^{-st}\, dt = \frac{1}{2}\int_{0^-}^{\infty} t^2 e^{-st}\, dt = \frac{1}{s^3}, \quad \sigma > 0$$

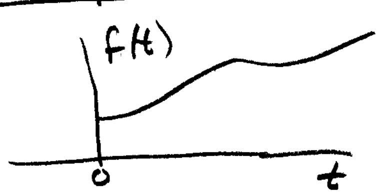
by parts, twice

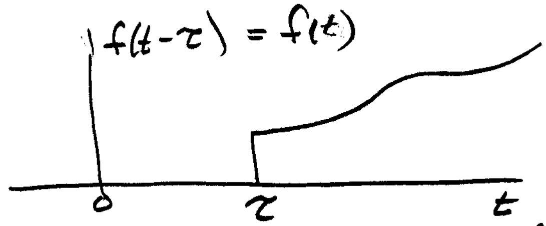






example delayed signal





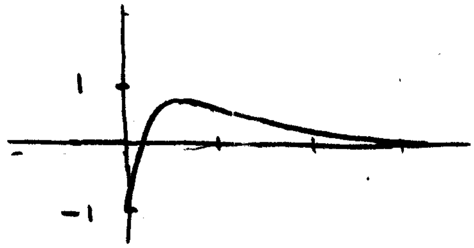
$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

$$\tilde{F}(s) = \int_{0^-}^{\infty} \tilde{f}(t)\, e^{-st}\, dt = \int_{\tau^-}^{\infty} f(t-\tau)\, e^{-st}\, dt$$

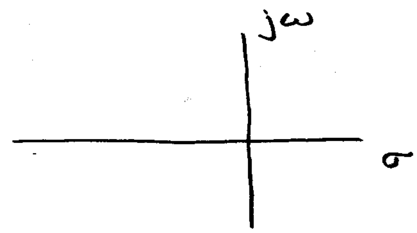
$$= \int_{0^-}^{\infty} f(\alpha)\, e^{-s(\alpha+\tau)}\, d\alpha$$

$$= e^{-s\tau} \int_{0^-}^{\infty} f(\alpha)\, e^{-s\alpha}\, d\alpha = e^{-s\tau} F(s)$$

example $f(t) = 2e^{-t} - 3e^{-4t}$



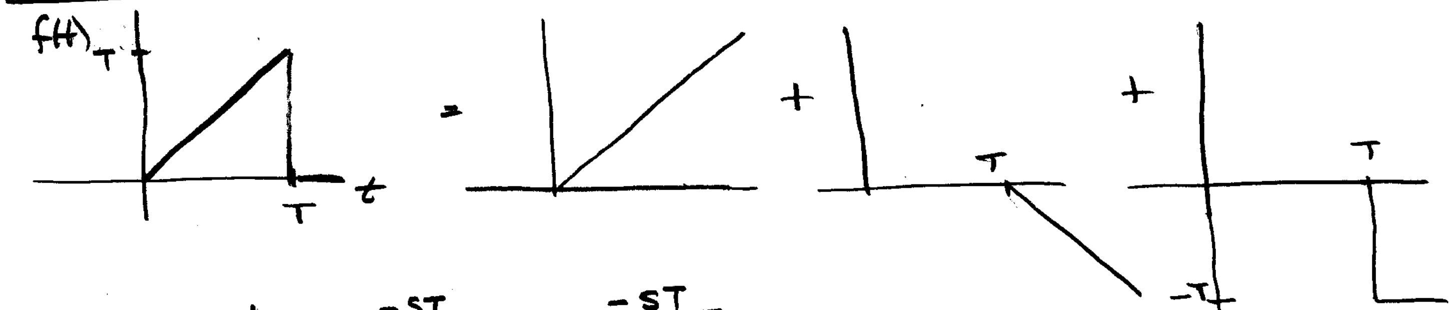
what is the ROC ?



$$F(s) = \frac{2}{s-{}^{-}1} - \frac{3}{s-{}^{-}4} = \frac{2}{s+1} - \frac{3}{s+4} = \frac{-s+5}{(s+1)(s+4)}$$

This suggests a way to invert rational polynomial transforms – expand to "partial fractions".

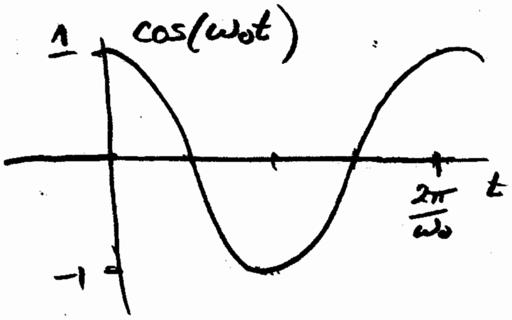
example 13.8 DC & L

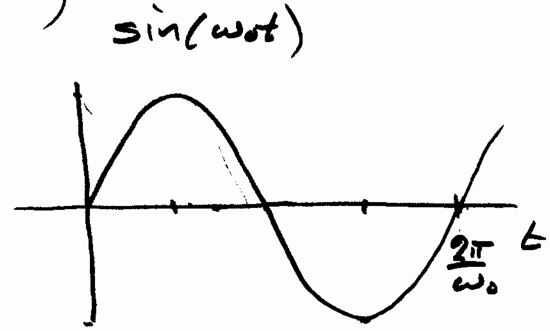


$$F(s) = \frac{1}{s^2} - e^{-sT}\frac{1}{s^2} - e^{-sT}\frac{T}{s}$$

$$= \frac{1}{s^2} - e^{-sT}\left(\frac{1+sT}{s^2}\right)$$

example $\cos(\omega_0 t)$ $\quad \sin(\omega_0 t)$





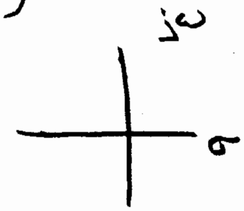
$$\int_{0^-}^{\infty} \cos(\omega_0 t)\, e^{-st}\, dt \qquad\qquad \int_{0^-}^{\infty} \sin(\omega_0 t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \tfrac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-st}\, dt \qquad\qquad = \int_{0^-}^{\infty} \tfrac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) e^{-st}\, dt$$

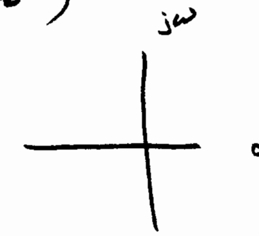
ROC?

$$= \frac{1}{2}\left(\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right) \qquad\qquad = \frac{1}{2j}\left(\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right)$$

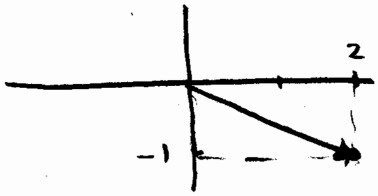
$$= \frac{s}{s^2+\omega_0^2}$$

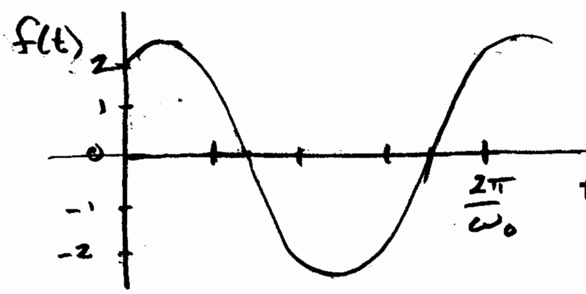


$$= \frac{\omega_0}{s^2+\omega_0^2}$$

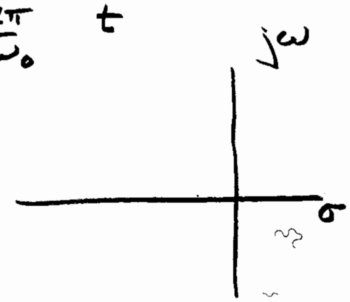


and with a phase shift: $f(t) = 2\cos(\omega_0 t) + \sin(\omega_0 t)$





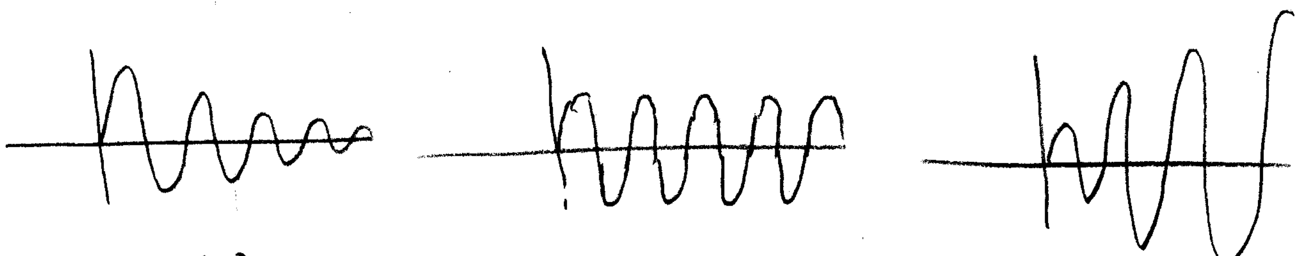
$$F(s) = \frac{2s + \omega_0}{s^2+\omega_0^2}$$



or as $f(t) = \sqrt{5}\cos\left(\omega_0 t - \tan^{-1}\left(\tfrac{1}{2}\right)\right)$

$$= \frac{\sqrt{5}}{2}\left(e^{-j\tan^{-1}(1/2)} e^{j\omega_0 t} + e^{j\tan^{-1}(1/2)} e^{-j\omega_0 t}\right)$$

- example $f(t) = e^{at}\sin(\omega_0 t)$



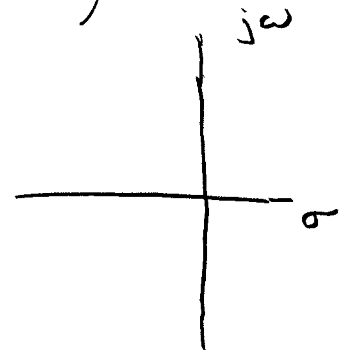
$a<0$ $\qquad a=0$ $\qquad a>0$

$$f(t) = e^{at}\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$= \frac{1}{2j}\left(e^{(a+j\omega_0)t} - e^{(a-j\omega_0)t}\right)$$

$$F(s) = \frac{1}{2j}\int_{0^-}^{\infty}\left(e^{(a+j\omega_0-s)t} - e^{(a-j\omega_0-s)t}\right)dt$$

$$= \frac{\omega_0}{(s-a)^2+\omega_0^2}$$



ROC ?

- For one-sided transforms, the ROC is to the right of the rightmost pole.