2.3 Inverting the Laplace Transform

- For the LT to be useful, we must be able to obtain the time function that corresponds to a given transform.

- Two methods are widely used for the rational polynomial transforms normally experienced in circuit analysis
  - The residue method springs from the theory of integral transforms and from complex variable theory. Easy to remember, sometimes cumbersome to use. See supplementary notes on LT.
  - The partial fractions method used in this course has more to remember, even more cumbersome to use. It's a general expansion, not necessarily linked to ILT
  - Actually, they are quite similar. Read DC+L for partial fractions, if you wish. I'll use residues.

- The partial fraction approach expresses the rational polynomial $F(s)$ as a sum of elementary fractions, each of which is a known LT. Invert by looking up a table.

ex

$$F(s) = \frac{3s^2+s+3}{(s+1)(s^2+4)} = \underbrace{\frac{1}{s+1}}_{\mathcal{L}^{-1}} + \underbrace{2\frac{s}{s^2+4}}_{\mathcal{L}^{-1}} \underbrace{- \frac{1}{2}\frac{2}{s^2+4}}_{\mathcal{L}^{-1}}$$

$$f(t) = e^{-t} + 2\cos(2t) - \tfrac{1}{2}\sin(2t)$$

$$f(t) = e^{-t} + 2\cos(2t) - \tfrac{1}{2}\sin(2t), \quad t \geq 0$$

or $\left(e^{-t} + 2\cos(2t) - \tfrac{1}{2}\sin(2t)\right)u(t)$

- We need a systematic way to expand such rational polynomials. There are two main cases:
  - distinct poles
  - repeated poles

If the poles are complex, it gets messy.

- $\boxed{\text{DC \& L Section 13.6}}$

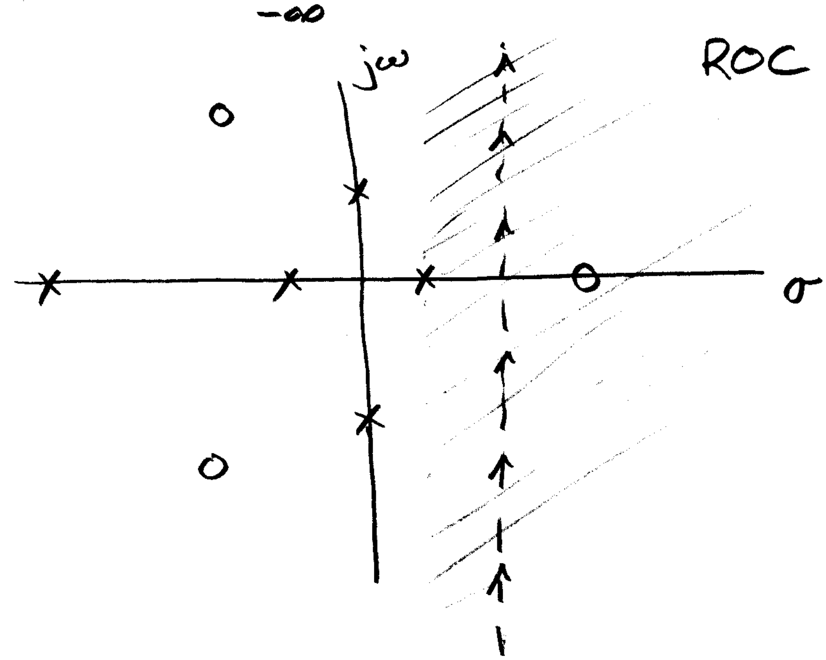
- We'll do the residue method. Before we get into purely mechanical procedures, we should link the ILT to an integral transform.
  - We want an inverse kernel $v(r,s)$ to give $f(r) = \int v(r,s) F(s)\, ds$.

    Use t, instead of r, since no confusion here,
  - ILT uses $v(t,s) = \frac{1}{2\pi j} e^{st}$. The transform variable $s = \sigma + j\omega$ is complex. How to integrate

$$\frac{1}{2\pi j} \int F(s)\, e^{st} ds \; ?$$

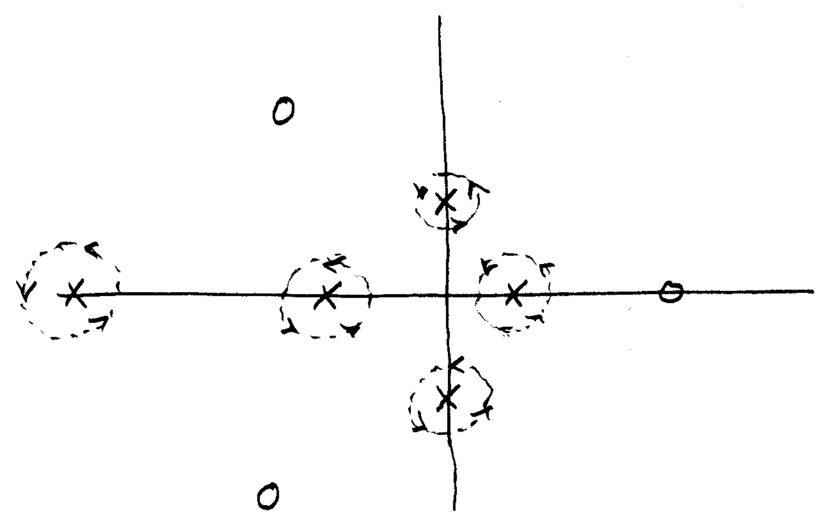
Answer: Choose σ to stay in ROC and vary ω. Then $ds = \sigma + j\, d\omega$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + j\omega)\, e^{(\sigma + j\omega)t} d\omega$$



More such background is in the supplementary notes.

- It can be shown with complex variable theory that, for rational polynomial transforms like ours (and many other form):
  - → The line integral equals the sum of integrals around each pole to the left of the line (for one sided)



  - → The integration contours are usually considered to be circular, for conceptual convenience only.
  - → Each circular integral gives a value termed the "residue" of the pole.
  - → We'll learn how to calculate residues.

- Inversion by residues.
  - The residue method performs the inversion without a prior expansion and without requiring you to look up a table.
  - The integrand is

$$F(s)\,e^{st} = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right) e^{st}}{(s-p_1)^{n_1}(s-p_2)^{n_2}\cdots(s-p_r)^{n_r}} \qquad m<n^{*},\ n=\sum_{i=1}^{r} n_i$$

Associated with each distinct pole is a residue $Res_i(t)$ which we'll learn to calculate. The ILT is the sum of the residues:

$$f(t) = \sum_{i=1}^{r} Res_i(t).$$

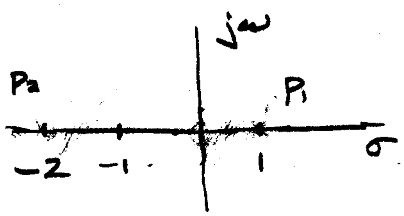
  - For a simple pole (multiplicity $n_i = 1$)

$$Res_i(t) = (s-p_i)F(s)\,e^{st}\Big|_{s=p_i}$$

Note that the effect of mult by $s-p_i$ is to remove the pole.

* We'll handle $m=n$ later (it's easy).

<u>example</u> $\quad F(s) = \dfrac{2s}{s^2+s-2} = \dfrac{2s}{\underset{P_1}{(s-1)}\underset{P_2}{(s+2)}}$



time constants?

$$Res_1(t) = (s-1)F(s)e^{st}\Big|_{s=1}$$

$$= \frac{2s\,e^{st}}{s+2}\Big|_{s=1} = \frac{2}{3}e^{t}$$

$$Res_2(t) = (s+2)F(s)e^{st}\Big|_{s=-2}$$

$$= \frac{2s}{s-1}e^{st}\Big|_{s=2} = \frac{4}{3}e^{-2t}$$

$$f(t) = \frac{2}{3}e^{t} + \frac{4}{3}e^{-2t} = \frac{2}{3}\left(e^{t} + 2e^{-2t}\right), \quad t \geq 0$$

– For a multiple pole

$$Res_i(t) = \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{ds^{n_i-1}}(s-p_i)^{n_i}F(s)e^{st}\Big|_{s=p_i}$$

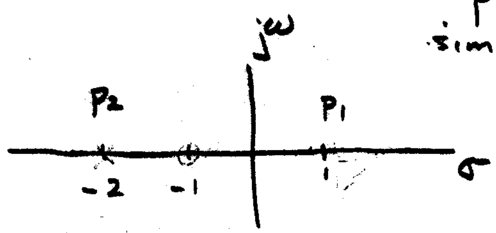
This gets very tedious if n_i is more than three or so and $F(s)$ is complicated.

example $\quad F(s) = \dfrac{s+1}{(s-1)(s+2)^2}$

$\qquad\qquad\qquad\qquad p_1 \qquad p_2$

$\qquad\qquad\qquad\quad$ simple $\quad$ double

time constants?



$f(t)$ is the sum of the two residues.

Pole p_1 is simple, so

$$\mathrm{Res}_1(t) = (s-1)\,F(s)\,e^{st}\Big|_{s=1} = \frac{(1+1)}{(1+2)^2}\,e^{t}$$

$$= \frac{2}{9}\,e^{t}$$

Pole p_2 is multiple (double)

$$\mathrm{Res}_2(t) = \frac{d}{ds}\,(s+2)^2 F(s)\,e^{st}\Big|_{s=-2}$$

$$= \frac{d}{ds}\,\frac{s+1}{s-1}\,e^{st}\Big|_{s=-2}$$

$$= \frac{\left(e^{st} + (s+1)\,t\,e^{st}\right)(s-1) - (s+1)\,e^{st}}{(s-1)^2}\Bigg|_{s=-2}$$

$$= \frac{-2e^{-2t} + 3te^{-2t}}{9}$$

Total:

$$f(t) = \frac{2}{9}e^{t} - \left(\frac{2}{9} - \frac{1}{3}t\right)e^{-2t}, \qquad t \geq 0$$

- Complex poles follow the same rules. The complex arithmetic can be messy, but you can cut the work in <u>half</u> by using the fact that the roots are in conjugate pairs. Here's how:

  - Think of $F(s)\,e^{st}$ as $\dfrac{D(s)\quad e^{st}}{(s-p_1)(s-p_1^*)}$

    where $D(s)$ is all of $F(s)$ but the complex poles.

  - Observe that the residues at p_1 and p_1^* are

$$\frac{D(p_1)e^{p_1 t}}{p_1 - p_1^*} \quad \text{and} \quad \frac{D(p_1^*)\,e^{p_1^* t}}{p_1^* - p_1},$$

or

$$\frac{D(p_1)e^{p_1 t}}{2j\,Im[p_1]} \quad \text{and} \quad \frac{\left(D(p_1)e^{p_1 t}\right)^*}{-2j\;Im[p_1]},$$

since coeffs in $D(s)$ are real and

$$e^{p_1^* t} = e^{Re[p_1]t}\,e^{-j\,Im[p_1]\,t}$$
$$= \left(e^{Re[p_1]t}\,e^{j\,Im[p_1]t}\right)^* = \left(e^{p_1 t}\right)^*$$

- Since the residues are also conjugate pairs, the contribution of p_1 and p_1^* to $f(t)$ is

$$\mathrm{Res}_{p_1}(t) + \mathrm{Res}_{p_1^*}(t) = \mathrm{Res}_{p_1}(t) + \mathrm{Res}^*_{p_1}(t)$$

$$= 2\,\mathrm{Re}\left[\mathrm{Res}_{p_1}(t)\right]$$

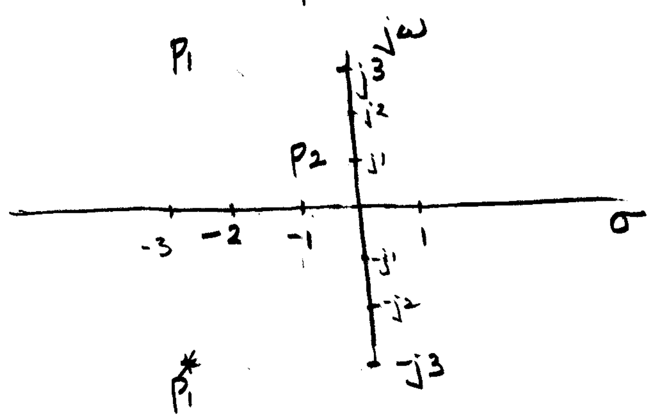
- so just evaluate the residue at <u>one</u> of the poles (either one) and take $2\,\mathrm{Re}\left[\mathrm{Res}_{p_1}(t)\right]$.

<u>example</u>

Invert $$F(s) = \frac{(s-1)}{(s+1)(s^2+4s+13)} = \frac{s-1}{\underset{p_2}{(s+1)}(s-p_1)(s-p_1^*)}$$

where $p_1 = -2 + j3$

time constants?



Do the easy one first.

$$\mathrm{Res}_2(t) = (s+1)F(s)e^{st}\Big|_{s=-1} = \frac{(s-1)e^{st}}{s^2+4s+13}\Big|_{s=-1}$$

$$= -\frac{1}{5}e^{-t}$$

Then one of the complex poles. Arbitrarily, $p_1 = -2+j3$,

$$\mathrm{Res}_{p_1}(t) = (s-p_1)F(s)e^{st}\Big|_{s=p_1}$$

$$= \frac{(s-1)e^{st}}{(s+1)(s-p_1^*)}\Big|_{s=p_1}$$

$$= \frac{(-3+j3)\,e^{(-2+j3)t}}{(-1+j3)\,j6}$$

Now simplify it by multiplying numerator and denominator by the conjugate of the denominator. This is the tedious part.

$$\mathrm{Res}_{p_1}(t) = e^{-2t}\frac{(-3+j3)(-1-j3)(-j6)\,e^{j3t}}{(-1+j3)(-1-j3)(j6)(-j6)}$$

$$= \frac{e^{-2t}}{10}(1-j2)e^{j3t}$$

The sum of the residues at p_1, p_1^* is

$$2\,\mathrm{Re}\left[\mathrm{Res}_{p_1}(t)\right] = \frac{e^{-2t}}{5}\,\mathrm{Re}\left[(1-j2)e^{j3t}\right]$$

We have alternatives for expressing the $Re\left[(1-j2)e^{j3t}\right]$: rectangular or polar.

<u>rectangular</u>

$$Re\left[(1-j2)(\cos(3t)+j\sin(3t))\right]$$

$$= \cos(3t) + 2\sin(3t)$$

<u>polar</u>

$$Re\left[\sqrt{5}\, e^{-j\tan^{-1}2} e^{j3t}\right]$$

$$= \sqrt{5}\cos(3t - \tan^{-1}(2))$$

Finally, the total solution:

$$f(t) = -\frac{1}{5}e^{-t} + \frac{e^{-2t}}{5}(\cos(3t) + 2\sin(3t)), \quad t \geq 0$$

or

$$f(t) = -\frac{1}{5}e^{-t} + \frac{e^{-2t}}{\sqrt{5}}\cos(3t - \tan^{-1}(2)), \quad t \geq 0$$

- Multiple complex poles require differentiation, exactly like multiple real poles, and they produce residues in complex conjugate pairs.

- And that's all there is to inverting Laplace transforms by residues...

... except for one more thing:

- What if $m = n$? Numerator, denominator have same degree.

  - Easy. You get an extra term,

    $K\,\delta(t)$

    where $K = \lim_{s \to \infty} F(s)$

  - example: $F(s) = \dfrac{2s^3 - s^2 - 6s + 12}{6(s+1)(s^2+9)}$

    $$f(t) = \frac{1}{3}\delta(t) + \underbrace{\sum_i \text{Res}_i(t)}_{\text{computed as usual.}}, \quad t \geq 0$$

  - example: $F(s) = \dfrac{3(s-2)}{s+1}$

    $$f(t) = 3\delta(t) + \text{Res}_1(t) = 3\delta(t) + 3(-3)e^{-t}, \quad t \geq 0$$

- Proof

→ $F(s) = \dfrac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0} = \dfrac{B(s)}{A(s)}$

→ Consider one step of synthetic division:

$$\begin{array}{r|l} & \quad b_n/a_n \\ a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0 & \; b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0 \\ & - \left(b_n s^n + \frac{a_{n-1}b_n}{a_n} s^{n-1} + \cdots + \frac{a_0 b_n}{a_n}\right) \\ \hline & \quad r_{n-1}s^{n-1} + \cdots + r_0 \end{array}$$

remainder $R(s)$

→ $F(s) = \dfrac{b_n}{a_n} + \dfrac{R(s)}{A(s)}$

$\mathcal{L}^{-1}$

$f(t) = \dfrac{b_n}{a_n}\delta(t) + \sum_i Res_i'(t)$

→ The new residues are the same as those of $F(s)$, since original residue

$$Res_i(t) = \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{ds^{n_i-1}}(s-p_i)^{n_i}F(s)e^{st}\Big|_{s=p_i}$$

$$= \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{ds^{n_i-1}}\Bigg(\underbrace{\frac{b_n}{a_n}(s-p_i)^{n_i}e^{st}}_{\text{this produces zero}} + \underbrace{(s-p_i)^{n_i}\frac{R(s)e^{st}}{A(s)}}_{\text{this produces } Res_i'(t)}\Bigg)\Bigg|_{s=p_i}$$

so $Res_i(t) = Res_i'(t)$

2.3.13

Some Questions

1. What time function $x(t)$ has $X(s) = 1$?

2. Suppose you have the transform $X(s)$ of some $x(t)$. What is the transform of the signal if it is delayed, $(\tilde{x}(t) = x(t-T))$?
   $\tilde{X}(s) = ?$

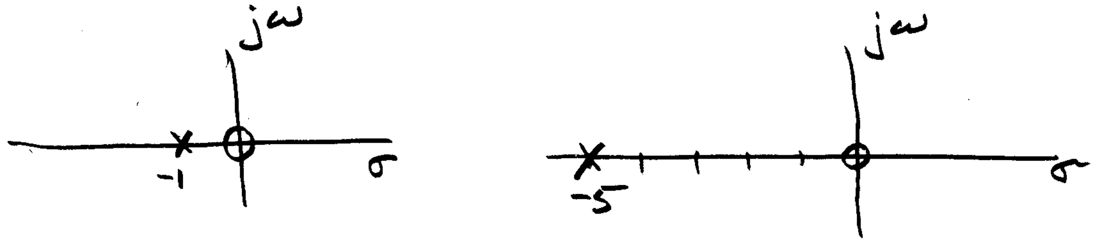
3. What is the region of convergence of
   $x(t) = e^{2t} \sin(3t+1)$?

4. What is the region of convergence of
   $x(t) = 2e^{-2t} + 5e^{3t} \quad t \geq 0$?

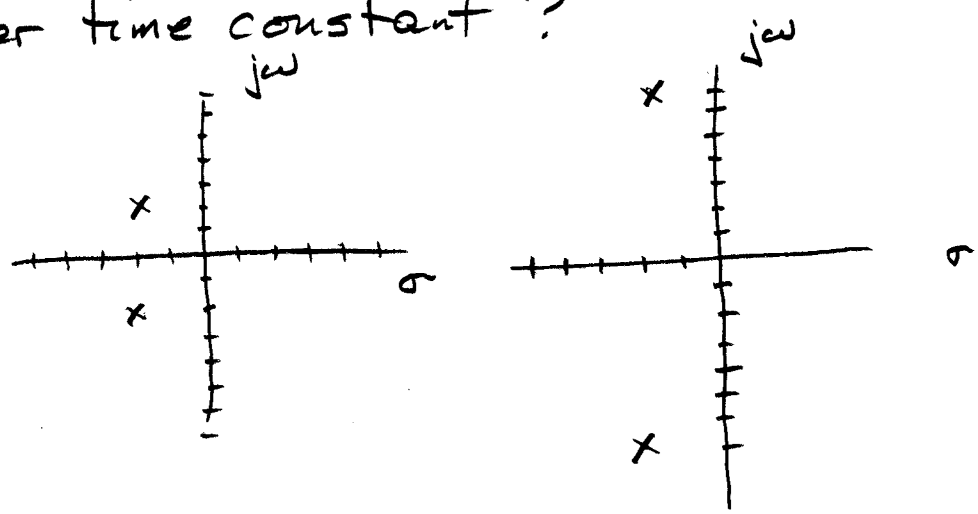
   $x(t) = 2te^{-2t} \quad t \geq 0$?

5. Invert $X(s) = \dfrac{4s}{s^2+s-6}$ to obtain $x(t)$.

6. Which pole-zero diagram corresponds to a time function with shorter time constant?



7. Which time function has more oscillations per time constant?



8. Invert $X(s) = \dfrac{s^2 - 4}{s^3 + 9s}$

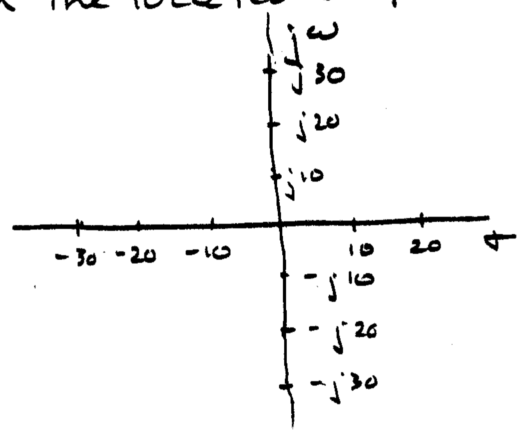
Question

Laplace inversion of a rational polynomial $F(s)$ produces a sum of time functions, each associated with a pole (or conjugate pair).

Suppose one such term is

$$3e^{-2t}\cos(30t - 1.2)$$

(a) Sketch the locations of the associated poles



(b) Give the values of

time const τ | damped natural freq ω_d | natural frequency ω_0 | damping factor ζ