2.4 Selected Properties of Laplace Transform

DC & L, 13.7

- In this section, we establish a few key properties of the LT, ones that make it so useful in circuit analysis and design.

- Multiplication/Convolution property.

This is why the LT is so widely used.

Suppose $y(t) = x(t) \circledast h(t) = \int_{\alpha} x(\alpha)\, h(t-\alpha)\, d\alpha$

what is its Laplace Transform $Y(s)$?

$$Y(s) = \int_{0^-}^{\infty} y(t)\, e^{-st}\, dt = \int_{0^-}^{\infty}\int_{-\infty}^{\infty} x(\alpha)\, h(t-\alpha)\, e^{-st}\, d\alpha\, dt$$

limits require $x(t)$, $h(t)$ defined as zero for $t<0$

$$= \int_{-\infty}^{\infty} x(\alpha)\left(\int_{0^-}^{\infty} h(t-\alpha)\, e^{-st}\, dt\right) d\alpha$$

$t = \beta + \alpha$

$$= \int_{-\infty}^{\infty} x(\alpha)\left(\int_{0^-}^{\infty} h(\beta)\, e^{-s(\alpha+\beta)}\, d\beta\right) d\alpha$$

$$= \int_{-\infty}^{\infty} x(\alpha)\, e^{-s\alpha}\left(\int_{0^-}^{\infty} h(\beta)\, e^{-s\beta}\, d\beta\right) d\alpha$$

$$= H(s) \int_{-\infty}^{\infty} x(\alpha)\, e^{-s\alpha}\, d\alpha$$

$$= H(s) \int_{0^-}^{\infty} x(\alpha)\, e^{-s\alpha}\, d\alpha \quad = \quad H(s)\, X(s)$$

- Summary: Convolution in the time domain corresponds to multiplication in the s domain: $Y(s) = H(s)\, X(s)$

$$\xrightarrow{x(t)} \boxed{h(t)} \xrightarrow{y(t) = h(t) \circledast x(t)}$$

$$\xrightarrow{X(s)} \boxed{H(s)} \xrightarrow{Y(s)} = H(s)\, X(s)$$

$H(s)$ is the "transfer function"

- This makes the strange commutative property of convolution a bit clearer:

$$\xrightarrow{H(s)} \boxed{X(s)} \xrightarrow{Y(s)} = X(s)\, H(s) = H(s)\, X(s)$$

or

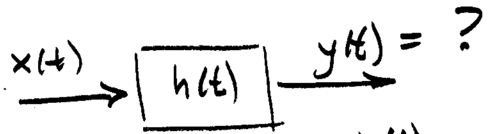
$$\xrightarrow{X(s)} \boxed{H_1(s)} \longrightarrow \boxed{H_2(s)} \xrightarrow{Y(s)} = H_2(s)\, H_1(s)\, X(s)$$

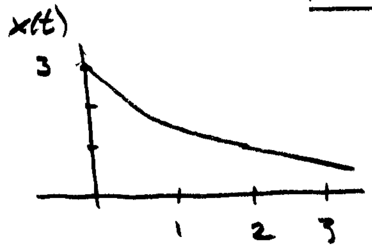
$$\xrightarrow{H_2(s)} \boxed{X(s)} \longrightarrow \boxed{H_1(s)} \xrightarrow{Y(s)} = H_1(s)\, X(s)\, H_2(s)$$

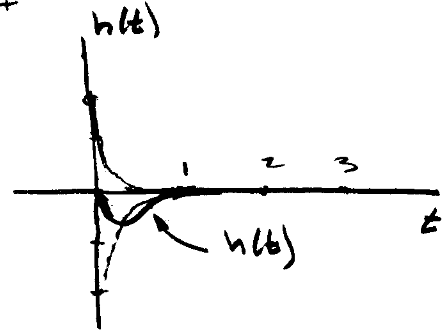
- An example of its use:

The input $x(t) = 3e^{-t/2}, \; t \geq 0$ is applied to a circuit with impulse response

$$h(t) = 2e^{-8t} - 2e^{-3t}, \quad t \geq 0$$







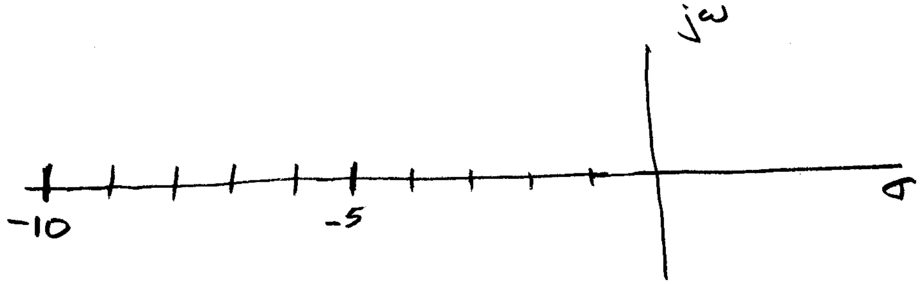
To find y(t), calculate X(s), H(s), Y(s), then $\mathcal{L}^{-1}$.

$$X(s) = \int_{0^-}^{\infty} 3e^{-t/2} e^{-st}\,dt = \frac{3}{s + 1/2}$$

$$H(s) = \int_{0^-}^{\infty} 2\left(e^{-8t} - e^{-3t}\right) e^{-st}\,dt = \frac{2}{s+8} - \frac{2}{s+3}$$

$$= \frac{-10}{(s+8)(s+3)}$$

$$Y(s) = H(s)\,X(s) = \frac{-30}{(s+8)(s+3)(s+\frac{1}{2})}$$



$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

poles $p_1 = -8$ $\quad p_2 = -3$ $\quad p_3 = -\frac{1}{2}$

$$Res_1(t) = \left.\frac{-30\, e^{st}}{(s+3)(s+\frac{1}{2})}\right|_{s=-8} = -\frac{4}{5} e^{-8t}$$

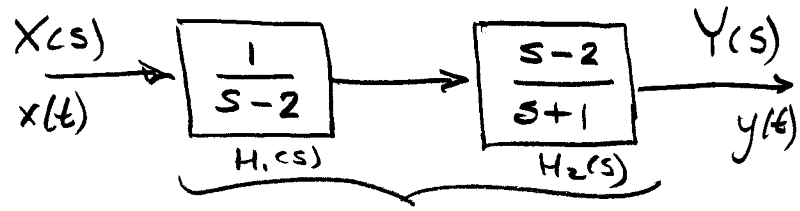
$$Res_2(t) = \left.\frac{-30\, e^{st}}{(s+8)(s+\frac{1}{2})}\right|_{s=-3} = \frac{12}{5} e^{-3t}$$

$$Res_3(t) = \left.\frac{-30\, e^{st}}{(s+8)(s+3)}\right|_{s=-\frac{1}{2}} = -\frac{8}{5} e^{-t/2}$$

$$y(t) = -\frac{4}{5} e^{-8t} + \frac{12}{5} e^{-3t} - \frac{8}{5} e^{-t/2}, \quad t \geq 0$$

- Another example. If you have determined the output $Y(s)$ corresponding to some test input $X(s)$, you can determine the internal dynamics of the circuit, as given by $H(s)$. (Some limitations, though).

- one limitation — pole-zero cancellation. Suppose two sub-circuits connected in cascade (series)



$$H(s) = H_1(s)\,H_2(s) = \frac{1}{s+1}$$

A test would give

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+1}$$

concealing the fact that subcircuit 1 is diverging through instability.

Less dramatic example $\frac{1}{s+2} \cdot \frac{s+2}{s+1} = \frac{1}{s+1}$

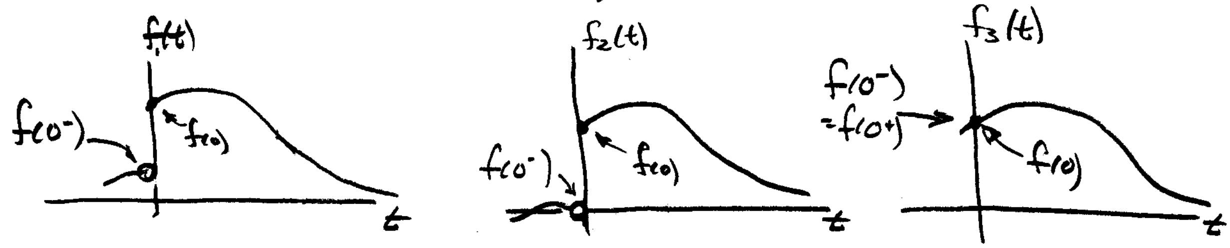
- another limitation: How do you calculate Y(s) from measured y(t)? How do you determine if rational polynomial is the best model? what degree, coeffs, of numerator and denominator?

- Time Differentiation Property

- This property is the key to DE solution by transforms, since it relates the transform of $\dot{f}(t)$ to that of $f(t)$.

  We'll obtain the result by sketches, then by the usual proof.

- First, reconsider the one-sided nature of our definition of LT. What if there are initial conditions $f(0^-)$?

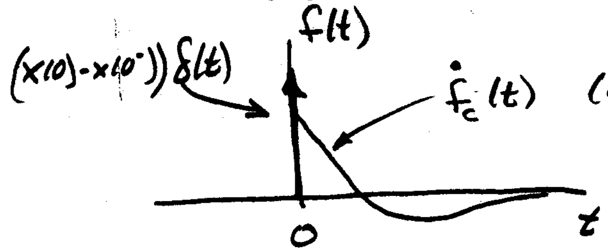


All these functions have the same LT

$$\int_{0^-}^{\infty} f(t)\,e^{-st}\,dt$$

since the contribution of the jump is infinitesimal.

- However, their derivatives have different LTs depending on the value of $f(0^-)$



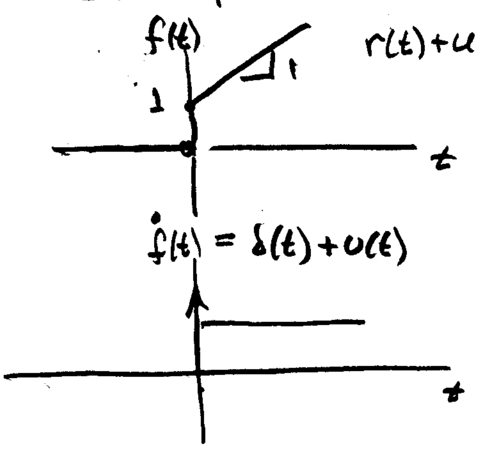
$$\mathcal{L}[\dot{f}(t)] = \int_{0^-}^{\infty} \dot{f}(t)e^{-st}dt$$

$$= \int_{0^-}^{0} (f(0)-f(0^-))\delta(t)\, e^{-st}dt + \int_{0}^{\infty} \dot{f}_c(t)e^{-st}dt$$

$$= f(0)-f(0^-) + \underbrace{\left(f(t)\, e^{-st}\Big|_0^\infty\right)}_{0\ -f(0)} + \underbrace{s\int_0^\infty f(t)\, e^{-st}dt}_{s\int_{0^-}^{\infty} f(t)e^{-st}dt}$$

$$= sF(s) - f(0^-)$$

- example:



$$F(s) = \mathcal{L}[r(t)+u(t)]$$
$$= \frac{1}{s^2}+\frac{1}{s} = \frac{s+1}{s^2}$$

$$\mathcal{L}[\dot{f}(t)] = \mathcal{L}[\delta(t)+u(t)]$$
$$= 1+\frac{1}{s} = \frac{s+1}{s}$$

and $= sF(s) - f(0^-) = sF(s)$

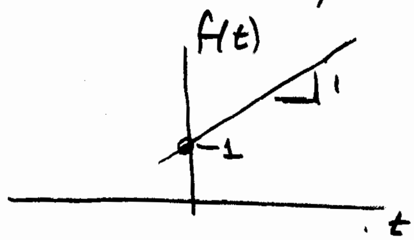
– Here's the direct proof without pictures:

$$\mathcal{L}[\dot{f}(t)] = \int_{0^-}^{\infty} \dot{f}(t)\, e^{-st}\, dt$$

$$= \left(f(t)e^{-st}\right|_{0^-}^{\infty} - \int_{0^-}^{\infty} -s f(t) e^{-st} dt$$

$$= s F(s) - f(0^-)$$

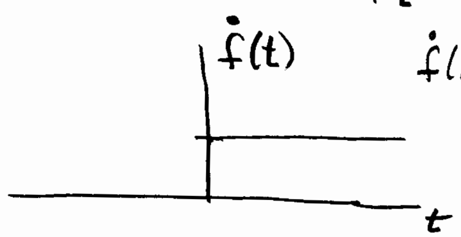
– example again – this time the ramp has no jump at time $t=0$, making $f(0^-) = f(0)$.



$f(t) = 1+t$

$$F(s) = \int_{0^-}^{\infty} (1+t) e^{-st} dt$$

$$= \frac{1}{s} + \frac{1}{s^2} = \frac{s+1}{s^2}$$



$\dot{f}(t) = 1$

as before.

The derivative and its transform have changed

$$\mathcal{L}[\dot{f}(t)] = \int_{0^-}^{\infty} 1\, e^{-st} dt$$

$$= \frac{1}{s}$$

but still $s F(s) - f(0^-)$

$$= s \frac{s+1}{s^2} - 1$$

$$= \frac{1}{s}$$

- Extending the differentiation property to higher order derivatives:

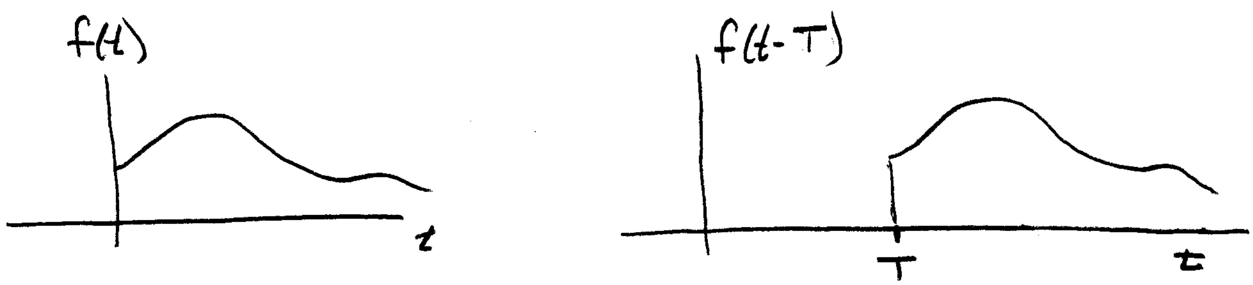
$$\mathcal{L}[\ddot{f}(t)] = s\,\mathcal{L}[\dot{f}(t)] - \dot{f}(0^-)$$
$$= s^2 F(s) - s f(0^-) - \dot{f}(0^-)$$

and

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f^{(1)}(0^-) - \cdots - f^{(n-1)}(0^-)$$

- Time Shift Property

- We have seen this one before, on p 2.2.5.



Given $F(s) = \mathcal{L}[f(t)]$, what is $\mathcal{L}[f(t-T)]$?

- Solution 1:

$$\mathcal{L}[f(t-T)] = \int_{T^-}^{\infty} f(t-T)\, e^{-st} dt = \int_{0^-}^{\infty} f(\alpha)\, e^{-s(T+\alpha)} d\alpha$$

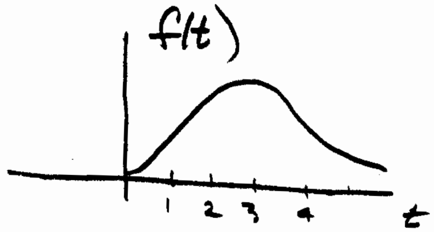
$$= e^{-sT} \int_{0^-}^{\infty} f(\alpha)\, e^{-s\alpha} d\alpha = e^{-sT} F(s)$$

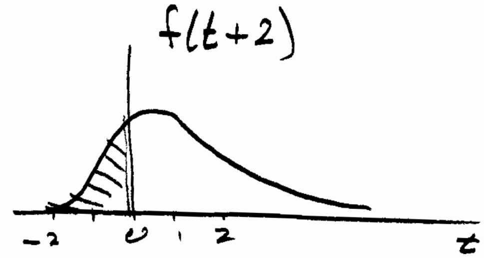
– Solution 2.

If $f(t) = \frac{1}{2\pi j} \int_L F(s) e^{st} ds$

then $f(t-T) = \frac{1}{2\pi j} \int_L F(s)\, e^{s(t-T)} ds = \frac{1}{2\pi j} \int_L \left(e^{-sT} F(s)\right) e^{st} ds$

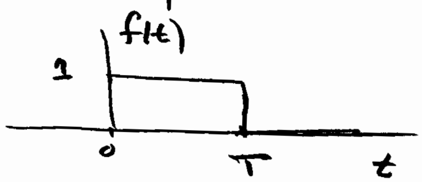
– Caution – if the net shift is a time advance, not delay, then property does not hold





$\mathcal{L}[f(t+2)] = \int_{0^-}^{\infty} f(t+2) e^{-st} dt$ loses part of the function.

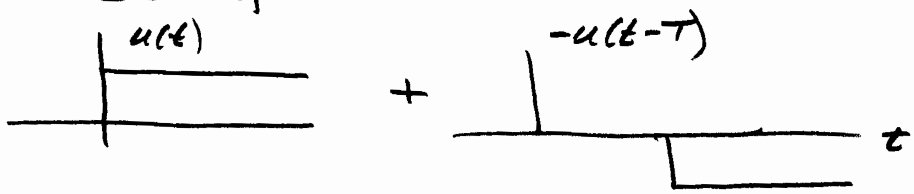
– example:



$F(s) = ?$

Direct route: $F(s) = \int_{0^-}^{T} 1 \cdot e^{-st} dt = \left(-\frac{1}{s} e^{-st}\right)\Big|_{0^-}^{T} = \frac{1}{s}\left(1 - e^{-sT}\right)$

Decomposition:



$\frac{1}{s} \qquad - e^{-sT} \cdot \frac{1}{s} \qquad = \frac{1}{s}\left(1 - e^{-sT}\right)$

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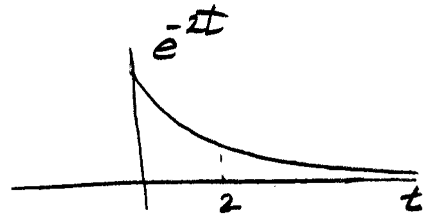
– example. If $F(s) = \dfrac{e^{-3s}}{s+2}$, what is $f(t)$?

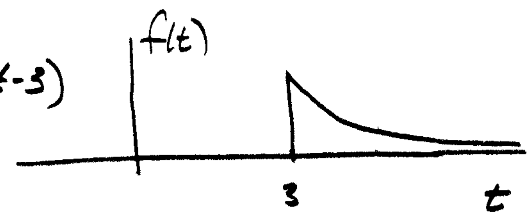
Hmmm — not a rational polynomial...

Factor out the e^{-3s}, recognizing it as a delay operator, invert the rest.

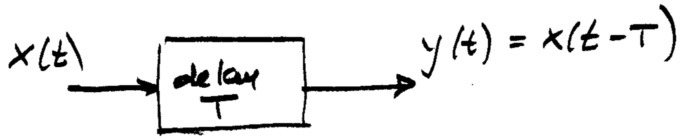
$$\frac{1}{s+2} \longleftrightarrow e^{-2t}$$



$$\frac{e^{-3s}}{s+2} \longleftrightarrow e^{-2(t-3)}u(t-3)$$



– example. Impulse response, transfer function of delayer?



impulse response: put in unit impulse, see what emerges

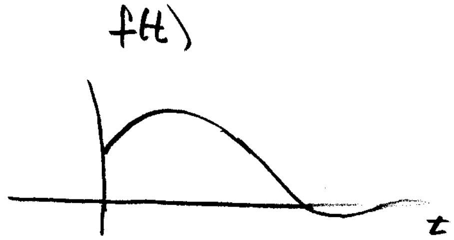
$$h(t) = \delta(t-T) \qquad y(t) = x(t) * h(t) = x(t-T)$$

transfer function $H(s) = \mathcal{L}[h(t)] = e^{-sT}$

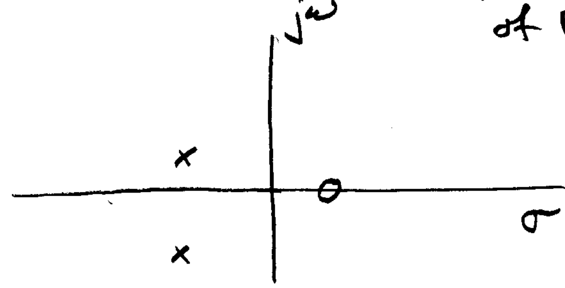
$$Y(s) = e^{-sT}X(s)$$

- **Time-Frequency Scaling Property**

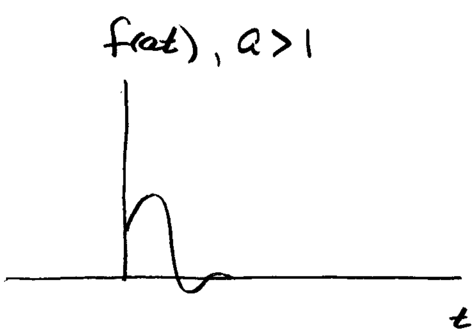
  - Very important — we often normalize time or frequency in filter design.



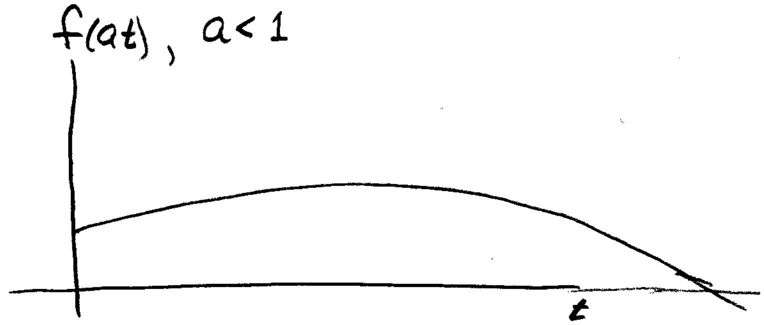
poles, zeros of F(s)



  - Consider $f(at)$, $a>0$



compression

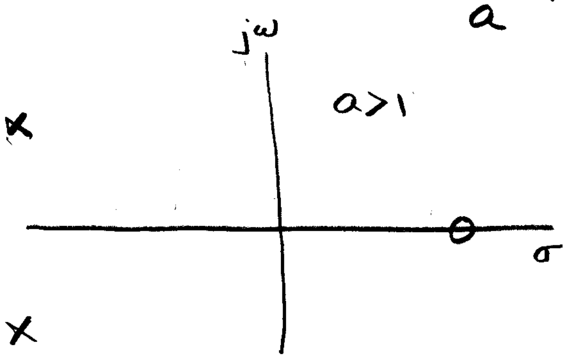


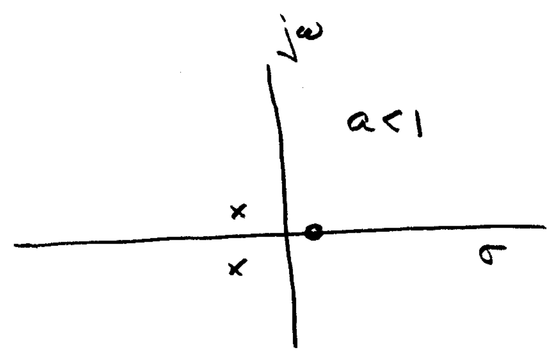
dilation

  - Effect on $\mathcal{L}[f(at)]$?

$$\mathcal{L}[f(at)] = \int_{0^-}^{\infty} f(at)\, e^{-st}\, dt = \int_{0^-}^{\infty} f(\beta)\, e^{-s\beta/a}\, \frac{d\beta}{a}$$

$$= \frac{1}{a} F(s/a)$$





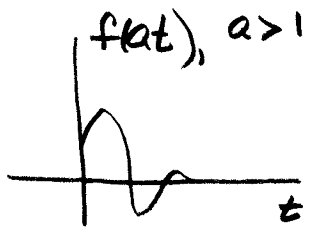
- Compression/dilation in one domain corresponds to dilation/compression in the other. And a scale factor.

- How to remember which way the scale factor goes?

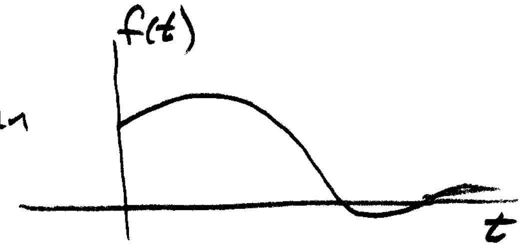
Just note $F(0) \left(= F(s)\big|_{s=0} \right)$

$$= \int_{0^-}^{\infty} f(t)\, e^{-0t}\, dt = \int_{0^-}^{\infty} f(t)\, dt = \text{area of } f(t).$$

for any transformable function.



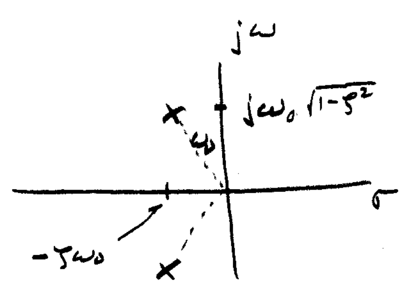
has less area than



so $\mathcal{L}[f(at)]$ is smaller: $\frac{1}{a} F(s/a)$.

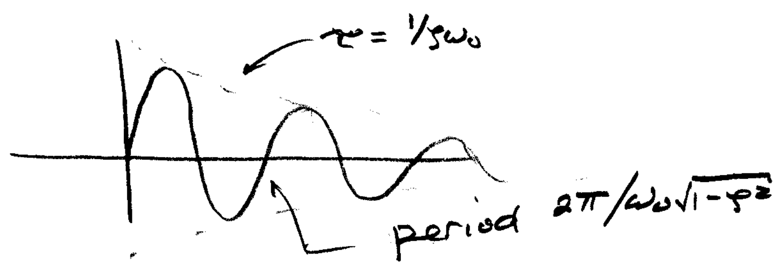
- example:

$$F(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$



and

$$f(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_0 t} \sin\left(\omega_0\sqrt{1-\zeta^2}\, t\right)$$



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Rewriting shows scaling by natural freq.

$$F(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$$

$$f(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}}\, e^{-\zeta(\omega_0 t)} \sin\left(\sqrt{1-\zeta^2}\,(\omega_0 t)\right)$$

which leads to a nice normalization:

$s \leftarrow \omega_0 u$ (u the new transform variable)

$$F(\omega_0 u) = \frac{1}{u^2 + 2\zeta u + 1}$$

which transforms to (using scaling property)

$$\frac{1}{\omega_0} f(t\omega_0) = \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\beta} \sin\left(\sqrt{1-\zeta^2}\,\beta\right)$$

where $t = \beta/\omega_0$ (β is the new scaled time)