2.5 Solving DEs by Laplace Transform

- Armed with the time differentiation property, we can now solve DEs using the LT.

- We'll start with an example, then the general form.

- example $\quad x(t) \rightarrow \boxed{\text{DE}} \rightarrow y(t)$

  - Suppose the circuit is described by

$$\dot{y} + 3y = x, \quad t \geq 0 \quad \text{with } y(0^-) = 1$$

(so it's not initially at rest). Response to $x(t) = u(t)$?

  - Transform the DE:

$$\mathcal{L}[\dot{y} + 3y] = \mathcal{L}[x(t)]$$

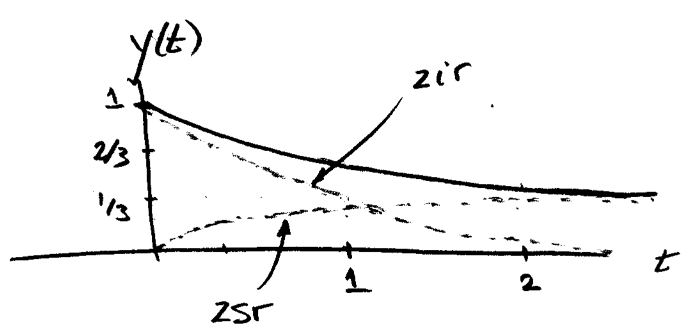
$$\mathcal{L}[\dot{y}] + 3\mathcal{L}[y] = \mathcal{L}[x(t)]$$

$$s\,Y(s) - 1 + 3\,Y(s) = X(s)$$

$$Y(s) = \underbrace{\frac{1}{s+3} X(s)}_{zsr} + \underbrace{\frac{1}{s+3}}_{zir} = \frac{1}{s(s+3)} + \frac{1}{s+3}$$

$$y(t) = \underbrace{\frac{1}{3} - \frac{e^{-3t}}{3}} + \underbrace{e^{-3t}}$$

$$= \frac{1}{3} + \frac{2}{3} e^{-3t}$$



- More generally, solve

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t)$$
$$= b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_0 x(t), \quad t \geq 0$$

with ICs $y^{(0)}(0^-) = c_0$, $y^{(1)}(0^-) = c_1$, $\cdots$, $y^{(n-1)}(0^-) = c_{n-1}$

– Transform the DE:

$$a_n\left(s^n Y(s) - s^{n-1}c_0 - s^{n-2}c_1 - \cdots - c_{n-1}\right)$$
$$+ a_{n-1}\left(s^{n-1} Y(s) - s^{n-2}c_0 - s^{n-3}c_1 - \cdots c_{n-2}\right)$$
$$+ \cdots + a_0 Y(s)$$
$$= b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \cdots + b_0 X(s) \quad \left(\begin{matrix}\text{note } x(t) \equiv 0 \\ \text{for } t < 0\end{matrix}\right)$$

– Collect terms

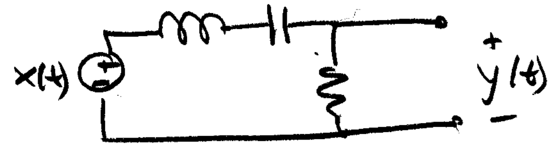
$$A(s) Y(s) = B(s) X(s) + a_n c_0 s^{n-1} + (a_n c_1 + a_{n-1} c_0) s^{n-2}$$
$$+ (a_n c_2 + a_{n-1} c_1 + a_{n-2} c_0) s^{n-3} + \cdots$$
$$\cdots + (a_n c_{n-1} + a_{n-1} c_{n-2} + \cdots + a_1 c_0)$$

$$A(s) Y(s) = B(s) X(s) + D(s)$$

$$Y(s) = \frac{B(s)}{A(s)} X(s) + \frac{D(s)}{A(s)}$$

$$= \underbrace{H(s) X(s)}_{\text{zsr}} + \underbrace{\frac{D(s)}{A(s)}}_{\text{zir}}$$

• example.



– A resonant circuit is described by

$$\ddot{y}(t) + 2\dot{y}(t) + 101\, y(t) = 4\dot{x}(t), \quad t \geq 0$$

and has values $y(0^-) = -5$, $\dot{y}(0^-) = 1$ just before application of the input

$$x(t) = \cos(8t), \quad t \geq 0$$

Find $y(t)$.

– Transform the DE

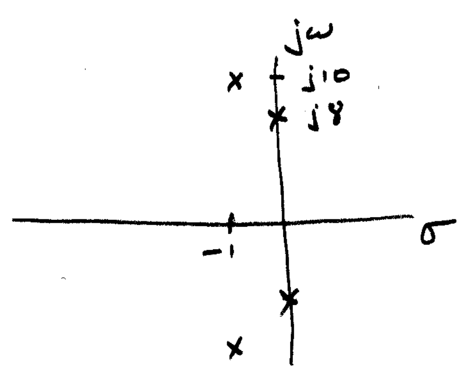
$$s^2 Y(s) - s\,y(0^-) - \dot{y}(0^-) + 2s\,Y(s) - 2y(0^-) + 101\,Y(s) = 4s\,X(s)$$

$$(s^2 + 2s + 101)\,Y(s) = 4s\,\frac{s}{s^2+64} + (-5)s + (1-10)$$

$$Y(s) = \underbrace{\frac{4s^2}{(s^2+2s+101)(s^2+64)}}_{zsr} - \underbrace{\frac{5s+9}{s^2+2s+101}}_{zir}$$

$$= -\frac{5s^3 + 5s^2 + 320s + 576}{(s^2+2s+101)(s^2+64)}$$

$p_1 = -1 + j10$ $\quad$ $p_2 = j8$



plus 3 zeros

$$\mathrm{Res}_{p_1}(t) = -\left.\frac{5s^3+5s^2+320s+576}{(s-p_1^*)(s^2+64)}e^{st}\right|_{s=-1+j10}$$

$$\equiv$$

$$= (-2.658 - j0.276)e^{-t}e^{j10t}$$

$$2\,\mathrm{Re}[\mathrm{Res}_{p_1}(t)] = \underbrace{-5.32\,e^{-t}\cos(10t) + 0.55\,e^{-t}\sin(10t)}_{\text{transient}}, \quad t \geq 0$$

and

$$\mathrm{Res}_{p_2}(t) = -\left.\frac{5s^3+5s^2+320s+576}{(s^2+2s+101)(s-p_2^*)}e^{st}\right|_{s=j8}$$

$$\equiv$$

$$= (0.158 + j0.364)e^{j8t}$$

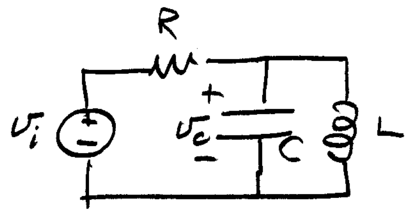
$$2\,\mathrm{Re}[\mathrm{Res}_{p_2}(t)] = \underbrace{0.315\cos(8t) - 0.729\sin(8t)}_{\text{steady state response}}, \quad t \geq 0$$

$$y(t) = -5.32\,e^{-t}\cos(10t) + 0.55\,e^{-t}\sin(10t) + 0.315\cos(8t) - 0.729\sin(8t), \quad t \geq 0$$

- Now we can obtain the transfer function H(s) just by reading the DE coefficients.

example from p. 1.2.2

$$\ddot{v}_c + \frac{1}{RC}\dot{v}_c + \frac{1}{LC}v_c = \frac{\dot{v}_i}{RC}$$



For zero state response

$$H(s) = \frac{V_c(s)}{V_i(s)} = \frac{s/RC}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

- And, of course, we can obtain the DE corresponding to a rational polynomial transfer function.

- Can we recover the DE from the impulse response h(t), assuming that it is a sum of terms, each (generally) of the form

$$h_i(t) = \{\text{poly}(t)\} \cdot \{e^{at}\} \cdot \{\cos(bt + \varphi)\}$$

$$h(t) = \sum_{i=1}^{N} h_i(t) \qquad ?$$

Yes: – Calculate

$$H(s) = \sum_{i=1}^{N} H_i(s),$$

each term a rational polynomial

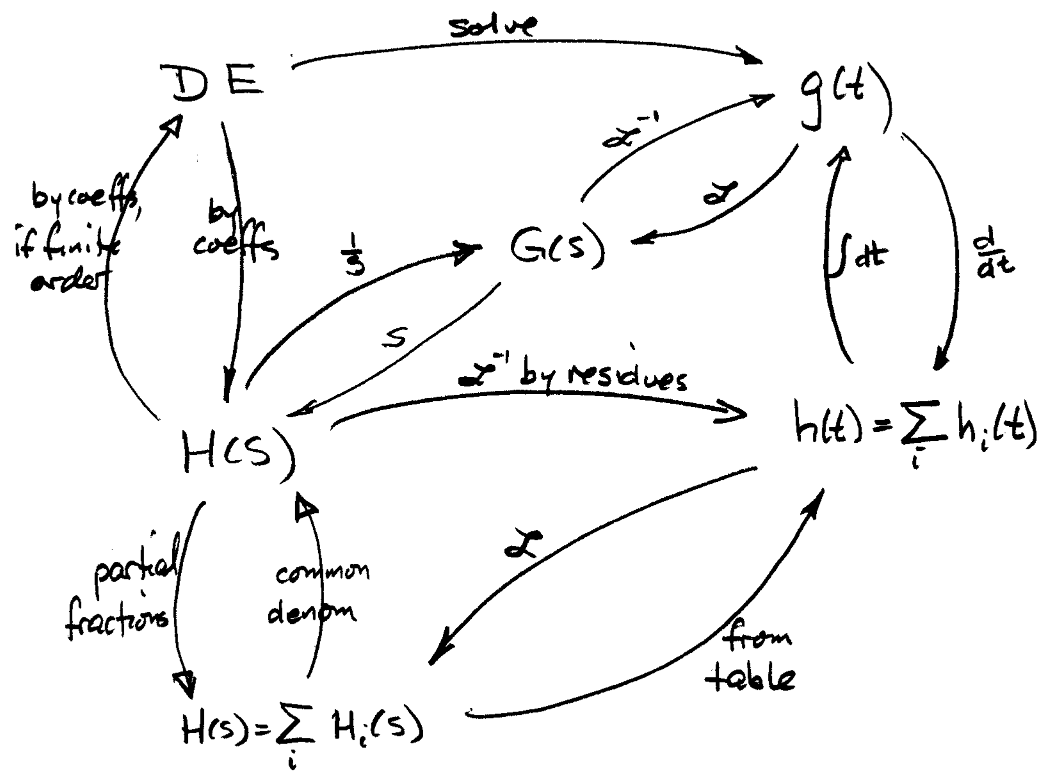
– Put over a common denominator

$$H(s) = \frac{b_m s^m + \cdots + b_0}{a_n s^n + \cdots + a_0}$$

– And the DE is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = b_m x^{(m)} + \cdots + b_0 x$$

- Connections:



The circuit descriptions are all equivalent.

- Connections among solution methods

$$x(t) \longrightarrow \boxed{} \longrightarrow y(t)$$

$$\begin{tikzcd}
x(t) \arrow[rr, bend left, "\text{solve DE}"] \arrow[rr, "x(t) \circledast h(t)"'] \arrow[d, bend right, "\mathcal{L}"'] & & y(t) \arrow[d, bend right, "\mathcal{L}"'] \\
X(s) \arrow[u, bend right, "\mathcal{L}^{-1}"'] \arrow[rr, "H(s)\,X(s)"] & & Y(s) \arrow[u, bend right, "\mathcal{L}^{-1}"'] \arrow[ll, bend left, "Y(s)/H(s)"]
\end{tikzcd}$$

- Effect of zeros

– Consider

$$a_n y^{(n)} + \cdots + a_0 y = b_m x^{(m)} + \cdots + b_0 x$$

– To interpret effect of derivatives on RHS, consider a "basic" solution to the simplified DE

$$a_n y_b^{(n)} + \cdots + a_0 y_b = x$$

The basic solution has time constants and natural frequencies determined by $A(s)$ and the input.

– By derivative property of LTI systems, solution of original DE is a linear comb of derivs of y_b

$$b_m y_b^{(m)} + \cdots + b_0 y_b = y$$

The effect of the derivs is to change amplitudes of terms & phase shifts, but not time constants and natural frequencies.

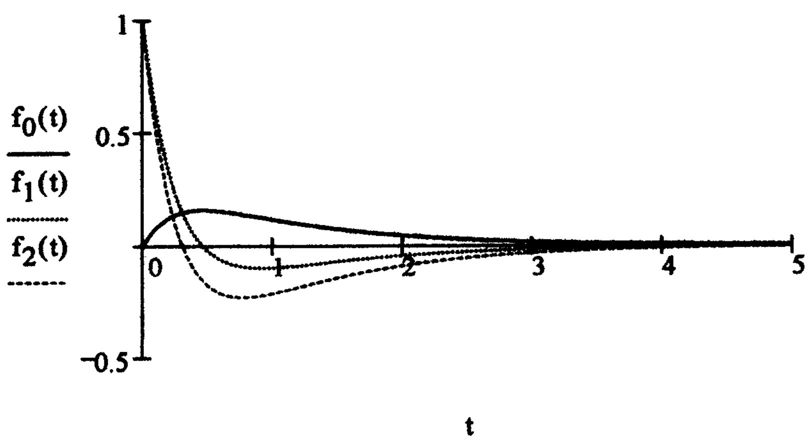
- Hence zeros – the roots of B(s) in

$H(s) = \frac{B(s)}{A(s)}$ — do not affect time consts or natural freqs.

$$F_0(s) := \frac{1}{(s+1)\cdot(s+4)} \qquad f_0(t) := \frac{1}{3}\cdot\exp(-t) - \frac{1}{3}\cdot\exp(-4\cdot t)$$

$$F_1(s) := \frac{s}{(s+1)\cdot(s+4)} \qquad f_1(t) := \frac{-1}{3}\cdot\exp(-t) + \frac{4}{3}\cdot\exp(-4\cdot t)$$

$$F_2(s) := \frac{s-1}{(s+1)\cdot(s+4)} \qquad f_2(t) := \frac{-2}{3}\cdot\exp(-t) + \frac{5}{3}\cdot\exp(-4\cdot t)$$



For reference, we'll look at the unit step response of a second order system without zeros.

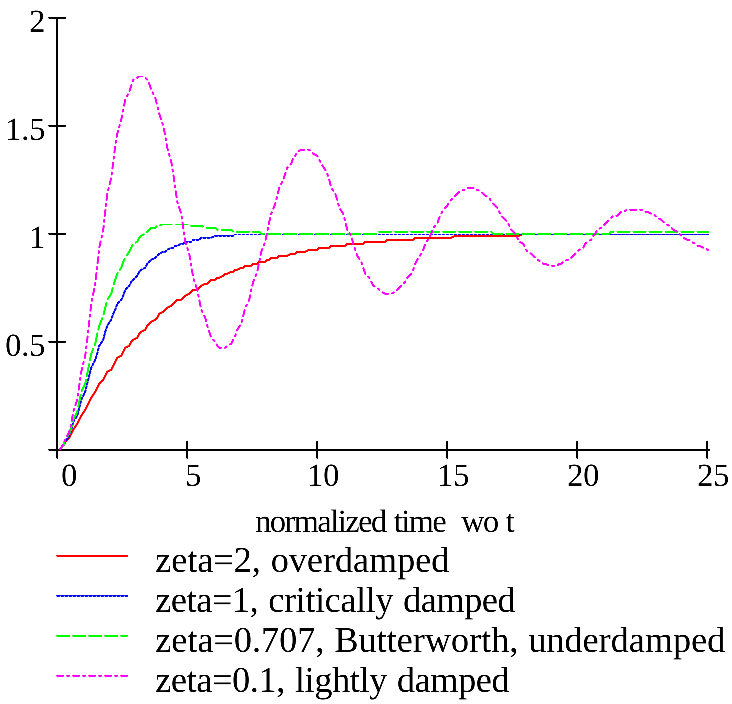
$$y^{(2)} + 2\cdot\zeta\cdot\omega_o\cdot y^{(1)} + \omega_o^{\,2}\cdot y = \omega_o^{\,2}\cdot x \qquad \text{for} \quad t \ge 0 \qquad \text{with} \qquad x(t) = u(t)$$

The step and impulse responses implicitly make the system initially at rest. The transfer function and the unit step response in the *s* domain are

$$H(s) = \frac{\omega_o^{\,2}}{s^2 + 2\cdot\zeta\cdot\omega_o\cdot s + \omega_o^{\,2}} \qquad \text{and} \qquad G(s) = \frac{\omega_o^{\,2}}{s\cdot\left(s^2 + 2\cdot\zeta\cdot\omega_o\cdot s + \omega_o^{\,2}\right)}$$

Standard inversion methods (residues or partial fractions) give

$$g(t) = 1 - e^{-\zeta\cdot\omega_o\cdot t}\cdot\left(\cos\left(\sqrt{1-\zeta^2}\cdot\omega_o\cdot t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\sin\left(\sqrt{1-\zeta^2}\cdot\omega_o\cdot t\right)\right)$$



Step Responses of 2nd Order Lowpass

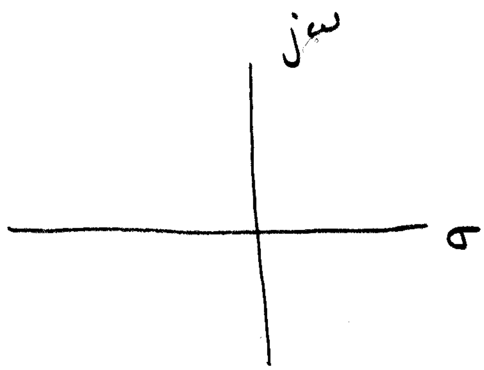
special form for ζ=1 (critically damped):

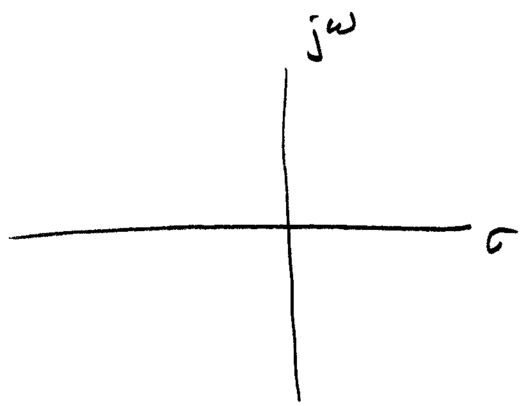
$$g(t) = 1 - \left(1 + \omega_o\cdot t\right)\cdot e^{-\omega_o\cdot t}$$

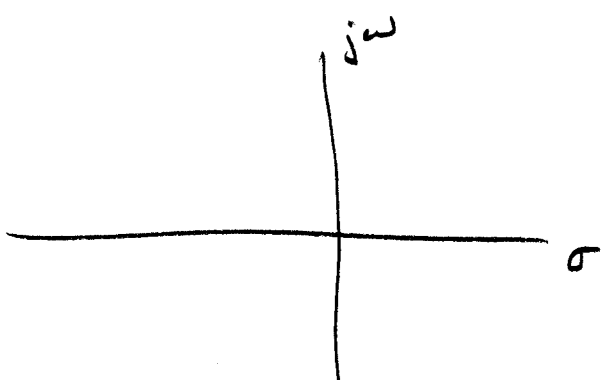
2.5.10

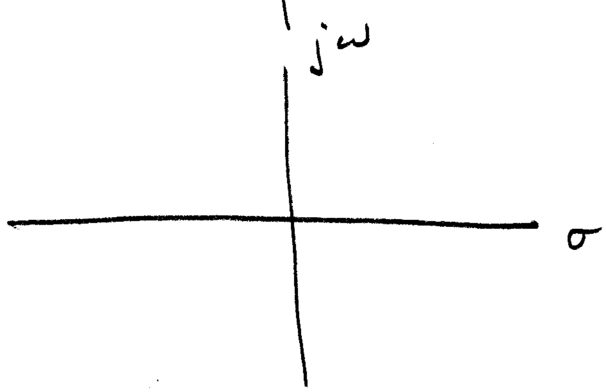
For these responses, note

- rise time
- overshoot
- pole zero diagram

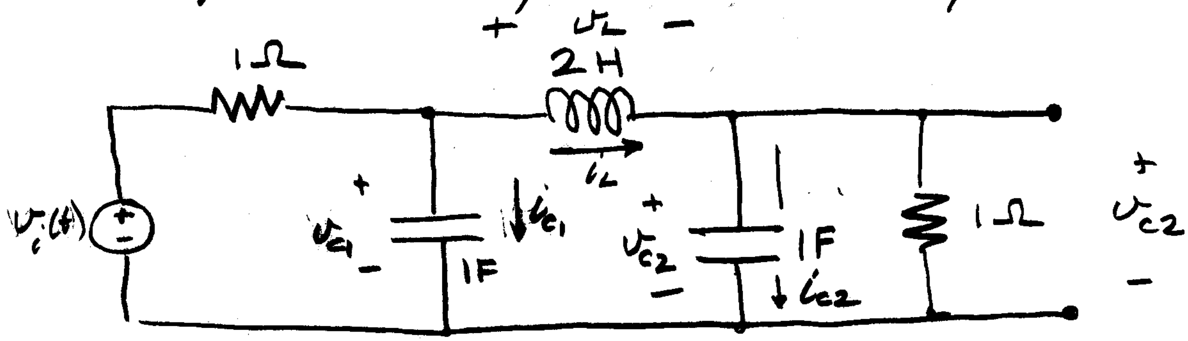








• Example – analyze a circuit by LT.



– This is a prototype "3rd order Butterworth" filter (more about this in Section 5).

For generality, allow nonzero ICs
$v_{c1}(0^-)$, $i_c(0^-)$, $v_{c2}(0^-)$.

– We want the transform $V_{c2}(s)$ and the transfer function $H(s) = V_{c2}(s)/V_i(s)$

Strategy: – get the state equations (as in Section 1.2.2)
– transform them
– solve.

– State equations. state vars are: ___ ___ ___

$$1F \cdot \dot{v}_{c1} = \frac{v_i - v_{c1}}{1\Omega} - i_L \qquad \dot{v}_{c1} = v_i - v_{c1} - i_L$$

$$2H\ \dot{i}_L = v_{c1} - v_{c2} \qquad 2\dot{i}_L = v_{c1} - v_{c2}$$

$$1F\ \dot{v}_{c2} = i_L - \frac{v_{c2}}{1\Omega} \qquad \dot{v}_{c2} = i_L - v_{c2}$$

all for $t \geq 0$

– Transform the state equations

$$s V_{c1}(s) - v_{c1}(0^-) = V_i(s) - V_{c1}(s) - I_L(s)$$

$$2s I_L(s) - 2 i_L(0^-) = V_{c1}(s) - V_{c2}(s)$$

$$s V_{c2}(s) - v_{c2}(0^-) = I_L(s) - V_{c2}(s)$$

and rearrange them

$$\begin{bmatrix} s+1 & 1 & 0 \\ -1 & 2s & 1 \\ 0 & -1 & s+1 \end{bmatrix} \begin{bmatrix} V_{c1}(s) \\ I_L(s) \\ V_{c2}(s) \end{bmatrix} = \begin{bmatrix} V_i(s) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} v_{c1}(0^-) \\ 2 i_L(0^-) \\ v_{c2}(0^-) \end{bmatrix}$$

– Solve for $V_{c2}(s)$

row 2 + $\frac{1}{s+1}$ row 1

$$\left[\begin{array}{ccc|c} s+1 & 1 & 0 & V_i + v_{c1}(0^-) \\ 0 & 2s+\frac{1}{s+1} & 1 & 2i_L(0^-) + \frac{V_i + v_{c1}(0^-)}{s+1} \\ 0 & -1 & s+1 & v_{c2}(0^-) \end{array}\right]$$

$$\frac{2s^2+2s+1}{s+1}$$

row 3 + $\frac{s+1}{2s^2+2s+1}$ row 2

$$\left[\begin{array}{ccc|c} s+1 & 1 & 0 & V_i + v_{c1}(0^-) \\ 0 & \frac{2s^2+2s+1}{s+1} & 1 & 2i_L(0^-) + \frac{V_i + v_{c1}(0^-)}{s+1} \\ 0 & 0 & s+1+\frac{s+1}{2s^2+2s+1} & v_{c2}(0^-) + \frac{2(s+1)}{2s^2+2s+1} i_L(0^-) + \frac{V_i + v_{c1}(0^-)}{2s^2+2s+1} \end{array}\right]$$

$$\frac{2s^3+4s^2+4s+2}{2s^2+2s+1}$$

Sol.

$$V_{c2}(s) = \frac{1}{2}\,\frac{1}{s^3+2s^2+2s+1}\cdot V_i(s)$$

$$+\frac{v_{c1}(0^-)}{2(s^3+2s^2+2s+1)}+\frac{2(s+1)\,i_L(0^-)}{2(s^3+2s^2+2s+1)}+\frac{(2s^2+2s+1)\,v_{c2}(0^-)}{2(s^3+2s^2+2s+1)}$$

Identify zsr, zir, $H(s)$.

Location of poles:

$$s^3+2s^2+2s+1 = (s+1)(s^2+s+1)$$

$$= (s+1)\left(s+\tfrac{1}{2}-j\tfrac{\sqrt{3}}{2}\right)\left(s+\tfrac{1}{2}+j\tfrac{\sqrt{3}}{2}\right)$$

poles are on the unit circle