2.6 More Properties of the Laplace Transform

- A few more useful properties, arranged in order of decreasing importance to ENSC 320

- Initial value and final value theorems

- These are very useful — easy answers to specific questions and a check on your work.

  → Given $F(s)$, find $f(0^+)$, $f(\infty)$ without doing a full inversion.

- Initial value theorem (IVT)

  We know $\mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\,dt = sF(s) - f(0^-)$

  Take limit as $s \to \infty$:

  Allowing step change $(f(0^+) \neq f(0^-))$ at $t=0$,

  $$LHS: \lim_{s\to\infty} \left[\int_{0^-}^{0^+} (f(0^+) - f(0^-))\,\delta(t)\, e^{-st}\,dt + \int_{0^+}^{\infty} \dot{f}(t) e^{-st}\,dt \right]$$

  $$= f(0^+) - f(0^-)$$

RHS : $\lim_{s\to\infty} sF(s) - f(0^-)$

so $f(0^+) - f(0^-) = \lim_{s\to\infty} sF(s) - f(0^-)$

$$\boxed{\lim_{s\to\infty} sF(s) = f(0^+)} \quad \text{IVT}$$

- Final value theorem (FVT)

Again $\int_{0^-}^{\infty} \frac{df}{dt} e^{-st} dt = sF(s) - f(0^-)$

Take limit as $s \to 0$:

$$f(\infty) - f(0^-) = \lim_{s\to 0} sF(s) - f(0^-)$$

$$\boxed{\lim_{s\to 0} sF(s) = f(\infty)} \quad \text{FVT}$$

- example:

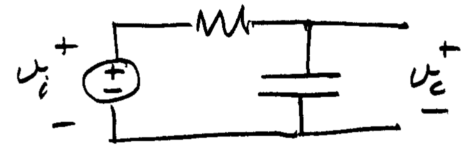
What is the initial value of $f(t)$ if

$$F(s) = \frac{s}{s^2-3s+2} \; ?$$

From IVT,

$$f(0^+) = \lim_{s\to\infty} sF(s) = 1$$

which is correct, since $f(t) = (2e^{2t} - e^{t})\, u(t)$.

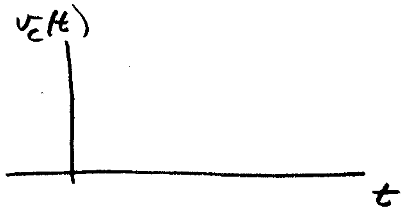
– example: 

$H(s) = \frac{1}{1+s\tau}$

(a) What are the initial and final values of $v_c(t)$ if $v_i(t)$ is a unit step?

$$V_c(s) = \frac{1}{s(1+s\tau)}$$

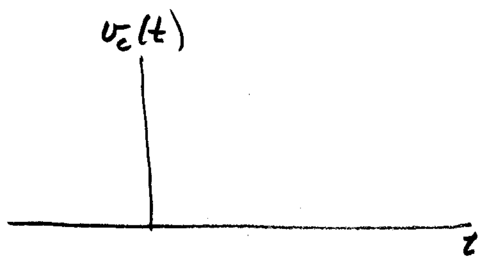
$v_c(0^+) = \lim_{s \to \infty} s V_c(s) =$ $\qquad$ $v_c(\infty) = \lim_{s \to 0} s V_c(s) =$



(b) and if $v_i(t)$ is a unit impulse?

$$V_c(s) = \frac{1}{1+s\tau}$$

$v_c(0^+) = \lim_{s \to \infty} s V_c(s) =$ $\qquad$ $v_c(\infty) = \lim_{s \to 0} s V_c(s) =$



- cautionary example:

Some functions may not have a well defined final value

$$f(t) = \cos(\omega_0 t) \qquad F(s) = \frac{s}{s^2+\omega_0^2} \qquad \lim_{s\to 0} sF(s) = 0$$

$$f(t) = e^{4t} \qquad F(s) = \frac{1}{s-4} \qquad \lim_{s\to 0} sF(s) = 0$$

or initial value

$$f(t) = \delta(t) - 5e^{-5t} \qquad F(s) = \frac{s}{s+5} \qquad \lim_{s\to\infty} sF(s) = \infty$$

- example:

What is the DC gain of a circuit with

$$H(s) = \frac{s+5}{s^3+3s^2+2s+1} \qquad ?$$

DC gain would be the steady state value of its response to a unit step input.

$$Y(s) = \frac{1}{s}\cdot H(s)$$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} H(s) = H(0) = 5$$

DC gain is $H(0)$!

(provided circuit is stable, so no poles in RHP; and our roots are -2.33, $-0.34 \pm j0.56$)

example: My circuit has

$$H(s) = \frac{s-2}{s^2+6s+25}$$

I calculated its step response as

$$g(t) = -\frac{2}{25} + e^{-3t}\left(-\frac{2}{25}\cos(4t) + \frac{31}{100}\sin(4t)\right).$$

Is this correct?

Try initial and final values.

$$G(s) = \frac{s-2}{s(s^2+6s+25)}$$

$\lim_{s\to\infty} sG(s) = 0$ but $g(0) = -\frac{4}{25}$

$\lim_{s\to 0} sG(s) = -\frac{2}{25}$ $g(\infty) = -\frac{2}{25}$

It's wrong.

Correction shows that the cos term should have been positive. Note that this test wouldn't find a sign error on the sin term.

- <u>Frequency differentiation property</u>

Start with

$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

Differentiate:

$$\frac{dF(s)}{ds} = -\int_{0^-}^{\infty} t f(t)\, e^{-st}\, dt$$

Hence

$$\boxed{\mathcal{L}[t f(t)] = -\frac{dF(s)}{ds}}$$

- Analogous to "differentiation in time corresponds to multiplication by s." The "dual."

- <u>Frequency shift property</u>

$$\mathcal{L}[e^{at} f(t)] = \int_{0^-}^{\infty} e^{at} f(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-(s-a)t}\, dt$$

$$= F(s-a)$$

- A dual of the time shift property

– example of use:

$$F(s) = \frac{s-2}{s^2+6s+25}$$

rewrite as

$$F(s) = \frac{(s+3)-5}{(s+3)^2+16}$$

so

$$f(t) = e^{-3t}\mathcal{L}^{-1}\left[\frac{\tilde{s}-5}{\tilde{s}^2+16}\right] = e^{-3t}\left(\cos(4t) - \frac{5}{4}\sin(4t)\right)u(t)$$

- <u>Convolution/multiplication property</u>

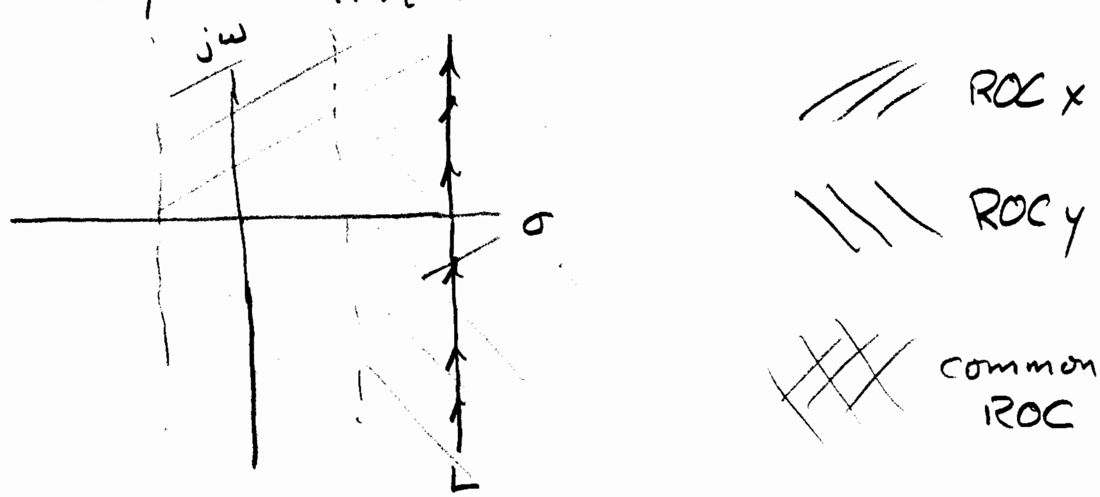
  – Another dual...

→ Multiplication in time corresponds to convolution in Laplace ("frequency") domain.

If $z(t) = x(t)\,y(t)$

then $Z(s) = \frac{1}{2\pi j}\int_L X(\nu)\,Y(s-\nu)\,d\nu$

with integration line L in the common ROC



We won't use this in our course.

- Series and parallel decompositions

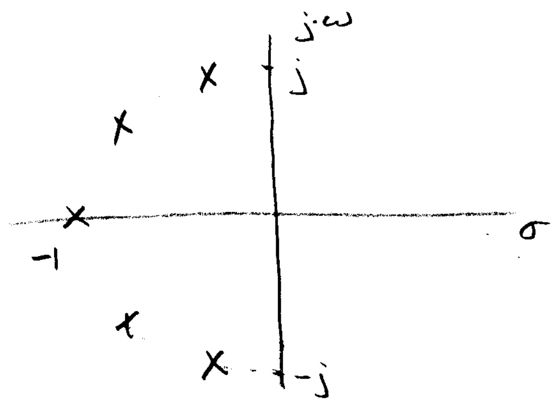
– Conceptually and practically it is sometimes useful to break a circuit into subcircuits.

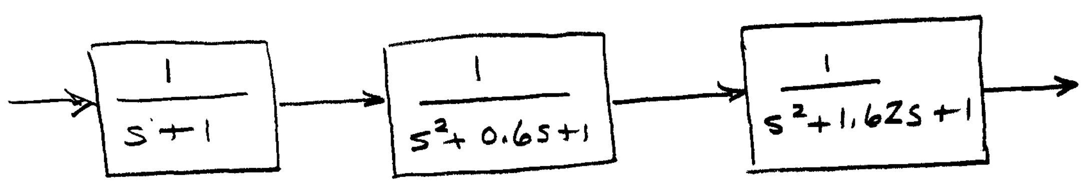
– Series (cascade) decomposition:

$$H(s) = \frac{1}{s^5 + 3.22s^4 + 5.192s^3 + 5.192s^2 + 3.22s + 1}$$

factors to

$$H(s) = \frac{1}{(s+1)(s^2+0.65s+1)(s^2+1.62s+1)}$$

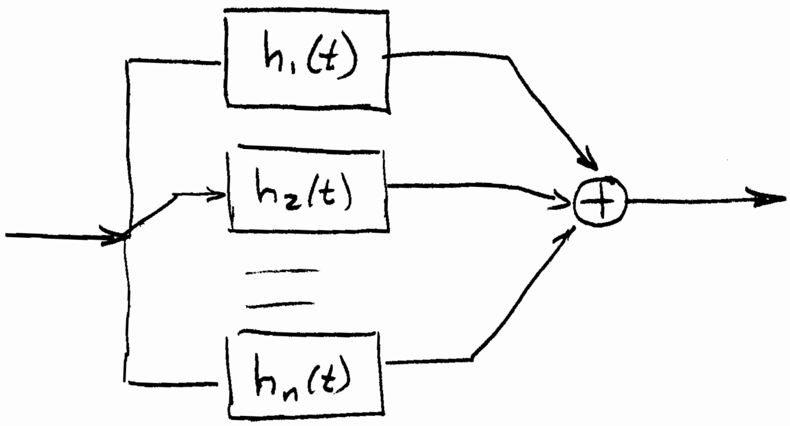




Doesn't have to be quad sections, but it reduces sensitivity to component error.

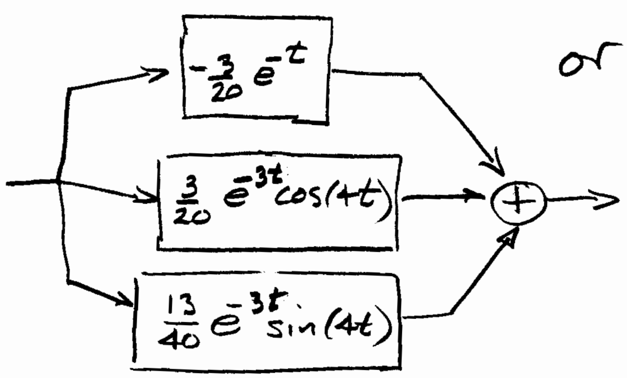
- parallel decomposition.

$$\begin{tikzcd} H(s) \arrow[r, "\mathcal{L}^{-1}"] \arrow[d, "\text{partial fractions}"'] & \sum_{i=1}^{n} h_i(t) \\ \sum_{i=1}^{n} H_i(s) & \end{tikzcd}$$



example $H(s) = \dfrac{s-2}{(s+1)(s^2+6s+25)}$

$$h(t) = -\frac{3}{20}e^{-t} + \frac{e^{-3t}}{40}\left(6\cos(4t) + 13\sin(4t)\right)$$



or