3. CIRCUIT ANALYSIS BY LAPLACE TRANSFORM

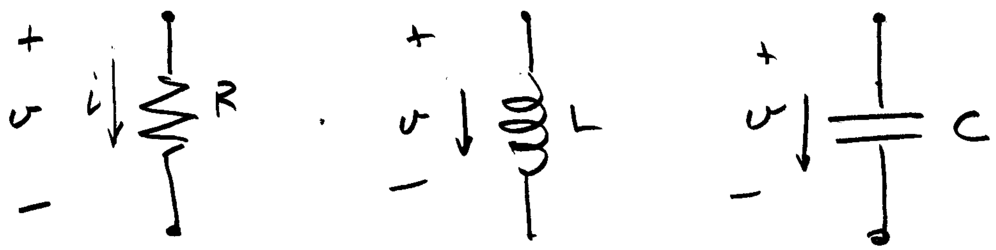
- By generalizing the ideas of resistance and conductance to impedance and admittance, we can analyze RLC circuits as easily as resistive circuits.

- Initial conditions are treated as equivalent sources.

- For linear circuits or subcircuits, it's an alternative to working through the DE.

3.1 Laplace Analysis of Circuits at Rest

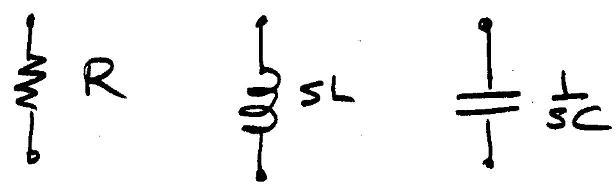
- Consider the terminal characteristics of circuit elements initially at rest (so the zero state response.



$v = iR \qquad v = L\frac{di}{dt} \qquad i = C\frac{dv}{dt}\,, \quad v = \frac{1}{C}\int_{0^-}^{t} i(\alpha)\,d\alpha$

$V(s) = R\,I(s) \qquad V(s) = sL\,I(s) \qquad I(s) = sC\,V(s) \quad V(s) = \frac{1}{sC}\,I(s)$

- The role of resistance is taken by a generalization, impedance, usually denoted $Z(s)$



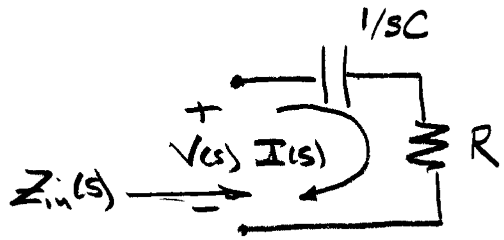
$Z_R = R \qquad Z_L = sL \qquad Z_C = \frac{1}{sC}$

– The role of conductance is taken by the generalization "admittance," usually denoted $Y(s)$

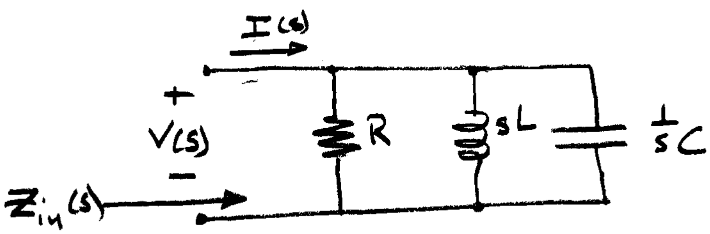
$$Y_R = G = \frac{1}{R} \qquad Y_L = \frac{1}{sL} \qquad Y_c = sC$$

- Because impedance relates the transforms $V(s)$, $I(s)$ just as resistance relates $v(t)$, $i(t)$ across a resistor, we can solve for transformed quantities just like solving a resistor network.

  – input (or driving point) impedance



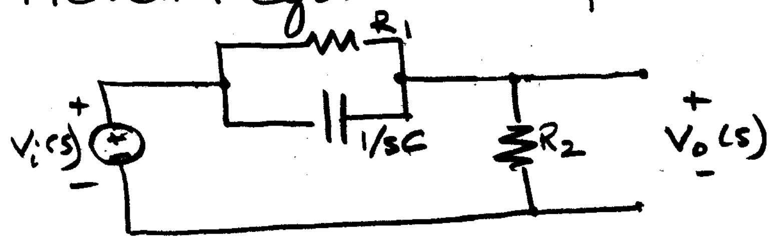
$$Z_{in}(s) = R + \frac{1}{sC} = \frac{V(s)}{I(s)}$$



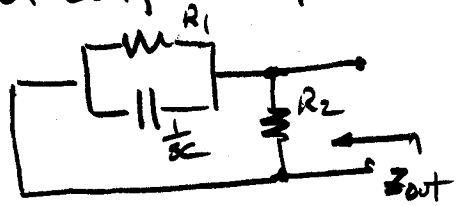
$$Z_{in}(s) = \left(\frac{1}{R} + \frac{1}{sL} + sC\right)^{-1} = \left(\frac{sL \cdot \frac{1}{sC} + R \cdot \frac{1}{sC} + RsL}{R \cdot sL \cdot \frac{1}{sC}}\right)^{-1}$$

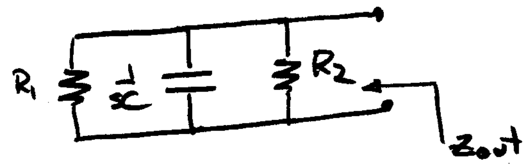
$$= \frac{s/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

– Thévenin equivalent of a source



→ for output impedance, set all sources to zero



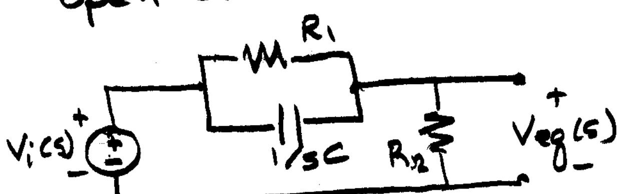


$$Z_{out}(s) = \frac{R_1 \cdot \frac{1}{sC} \cdot R_2}{R_2 \cdot \frac{1}{sC} + R_1 R_2 + R_1 \cdot \frac{1}{sC}} = \frac{1/C}{s + \frac{R_1 + R_2}{R_1 R_2 C}}$$

$$= \frac{1/C}{s + \frac{1}{R_{\parallel} C}} \qquad R_{\parallel} = R_1 \parallel R_2$$

(what if it had been a current source?)

→ for equivalent source voltage, keep output open circuited and calculate output voltage.

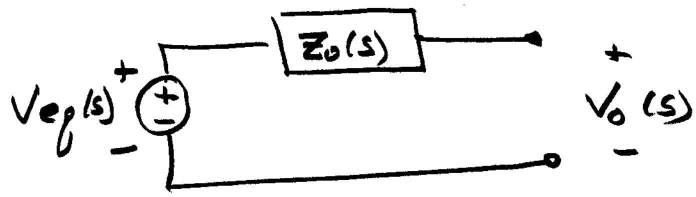


voltage divider

$$V_{eq}(s) = \frac{R_2}{R_2 + \frac{R_1/sC}{R_1 + \frac{1}{sC}}} V_i(s) = \frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_{\parallel} C}} \cdot V_i(s)$$

$R_{\parallel} = R_1 \parallel R_2$

→ Thévenin equivalent:



– Special terms for specific transfer functions

→ "transfer impedance": input is current source, designated output is some voltage

$$V(s) = H(s)\,I(s) \qquad H(s) \text{ is a } Z(s)$$

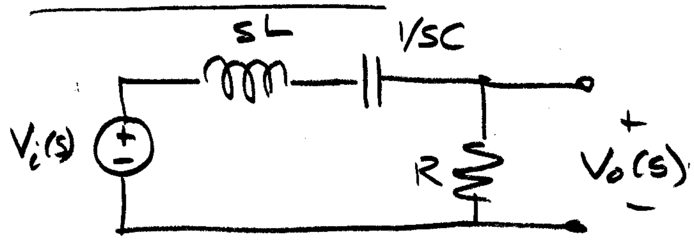
→ "transfer admittance": input is voltage source, designated output is a current:

$$I(s) = H(s)\,V(s) \qquad H(s) \text{ is a } Y(s).$$

Generally, though, we'll just use "transfer function".

• The rest of this section is just examples.

Series RLC



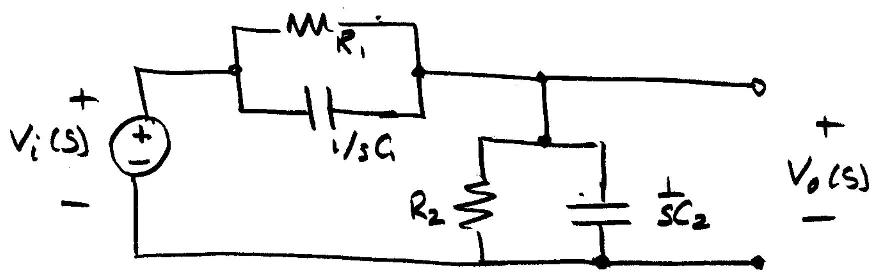
– Find $H(s)$ for input $V_i(s)$, output $V_o(s)$.

- Use voltage divider:

$$V_o(s) = V_i(s) \frac{R}{sL + \frac{1}{sC} + R} = \underbrace{\frac{s \cdot R/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}}_{H(s)} V_i(s)$$

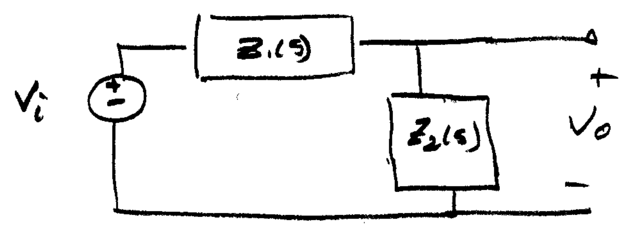
Easy to get the DE (if you want it)

Lead-lag (or lag-lead).



(a) Find $V_o(s)$.

- Voltage divider



$$V_o(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} V_i(s)$$

$$Z_1(s) = \frac{R_1/sC_1}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{1 + sR_1C_1}$$

$$= \frac{\frac{R_2}{1 + sR_2C_2}}{\frac{R_1}{1 + sR_1C_1} + \frac{R_2}{1 + sR_2C_2}} V_i(s)$$

$$= \underbrace{\frac{C_1}{C_1 + C_2} \cdot \frac{s + 1/R_1C_1}{s + \frac{R_1 + R_2}{R_1R_2(C_1 + C_2)}}}_{H(s)} \cdot V_i(s)$$

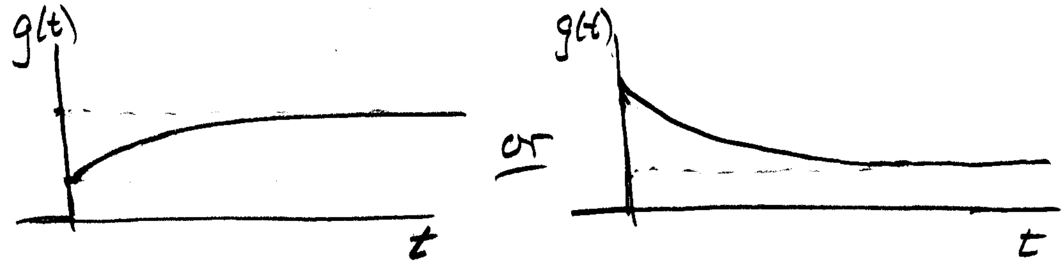
– So one pole, one zero, both negative real.

Time constant $\tau = R_{||}(C_1+C_2)$

(b) Step response?

initial value $\lim_{s\to\infty} s \cdot H(s) \cdot \frac{1}{s} = \frac{C_1}{C_1+C_2}$

final value $\lim_{s\to 0} s \cdot H(s) \frac{1}{s} = \frac{R_2}{R_1+R_2}$



(c) How much design freedom in setting pole, zero locations and DC gain?

gain $= \frac{R_2}{R_1+R_2}$ constrains the ratio $R_2/R_1 = \frac{g}{1-g}$

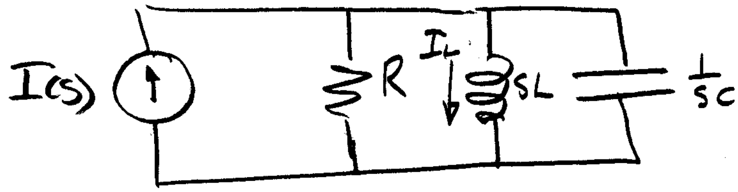
$p = -\frac{1}{R_{||}(C_1+C_2)}$ we have $R_{||} = \frac{R_2 R_1}{R_2+R_1} = g R_1$, so any value still possible.

$R_1 = \frac{1}{g(-p)(C_1+C_2)}$

$z = -\frac{1}{R_1 C_1} = \frac{g(C_1+C_2)}{C_1} p = g\left(1+\frac{C_2}{C_1}\right) p$

We have $g<1$, $1+C_2/C_1 > 1$, so trade them off for $z < p$ or $z > p$.

Current Divider



What is $I_L(s)$?

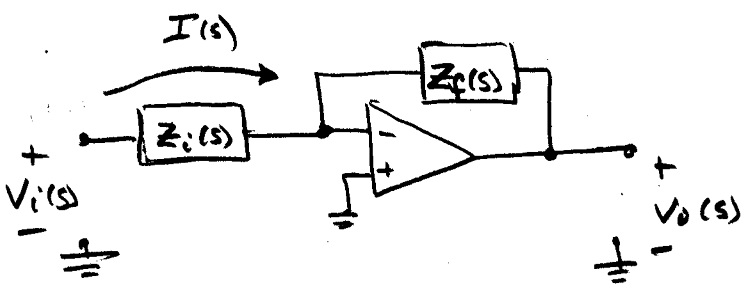
Use current divider:

$$I_L(s) = \frac{Y_L}{Y_R + Y_L + Y_C} I(s) = \frac{\frac{1}{sL}}{\frac{1}{R} + \frac{1}{sL} + sC} I(s)$$

$$= \frac{1/LC}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} I(s)$$

Ideal Op Amp with Feedback

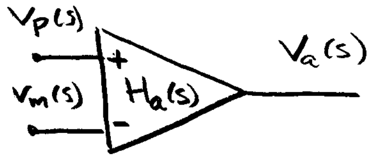
$\boxed{\text{DC\&L ex 14.5}}$



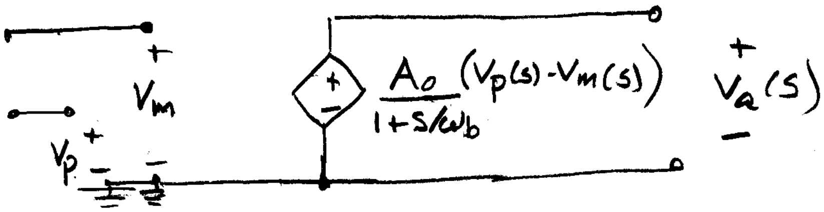
Simple model: negative feedback, so virtual ground at "−" input; KCL at "−" input makes

$$I(s) = \frac{V_i(s)}{Z_i(s)} \qquad V_o(s) = -Z_f(s)I(s) = -\frac{Z_f(s)}{Z_i(s)} V_i(s)$$

Non-Ideal Op-Amp

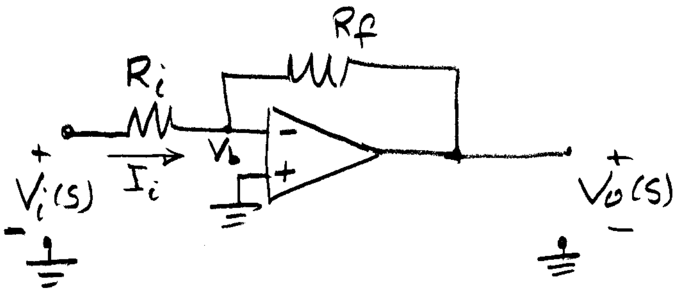


Gain is <u>very</u> high for low frequencies, falls off as $1/\omega$ for high frequencies. Model as 1^{st} order low pass: $H_a(s) = A_0/(1+s/\omega_b)$



still idealized input, output impedances

– How does this affect the usual amplifier?



$$I_i(s) = \frac{V_i(s) - V_o(s)}{R_i + R_f}$$

so

$$V_b(s) = V_i(s) - R_i \frac{V_i(s) - V_o(s)}{R_i + R_f} = \frac{R_f}{R_i + R_f} V_i(s) + \frac{R_i}{R_i + R_f} V_o(s)$$

and

$$V_o(s) = -H_a(s)\, V_b(s)$$

Solution
$$V_o(s) = \frac{-H_a(s)\,\frac{R_f}{R_i+R_f}}{1+H_a(s)\,\frac{R_i}{R_i+R_f}} \cdot V_i(s)$$

and if $H_a(s)$ is just a large gain, $V_o(s) \approx -\frac{R_f}{R_i} V_i(s)$

But substitute for $H_a(s)$, and get (after tidying)

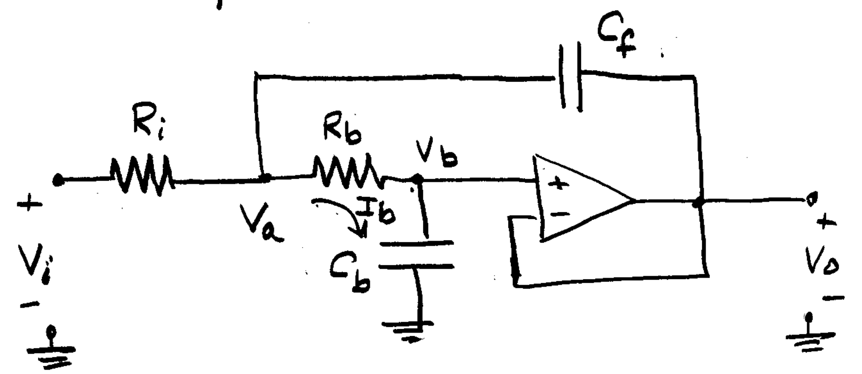
$$V_o(s) = -G \cdot \frac{A_o}{A_o+1+G} \cdot \frac{1}{1+s\cdot\frac{1+G}{\omega_b(1+G+A_o)}} \cdot V_i(s)$$

where $G = R_f/R_i$

An improvement in time constant, as well as gain determined by R_f/R_i (and insensitive to A_o). However, ambitious (i.e. large) values of G will increase the time constant, making the overall circuit slower to respond.

Sallen and Key Active Lowpass

- DC & L example 14.11



$$\frac{V_o(s)}{V_i(s)} = ?$$

- With an ideal op amp and negative feedback, voltage diff across input terminals is zero, so

$$V_b(s) = V_o(s)$$

- DC & L follow this observation with a classical nodal analysis at V_a and V_b. Read and understand.

- We'll take an ad hoc shortcut:

  - If $V_b = V_o$, then

$$I_b(s) = \frac{V_o(s)}{1/sC_b} = V_o(s)\, sC_b$$

  - which makes

$$V_a = R_b I_b + V_o = (1 + sR_bC_b)\, V_o$$

- so we can relate $V_i(s)$ and $V_o(s)$ through the node equation at V_b:

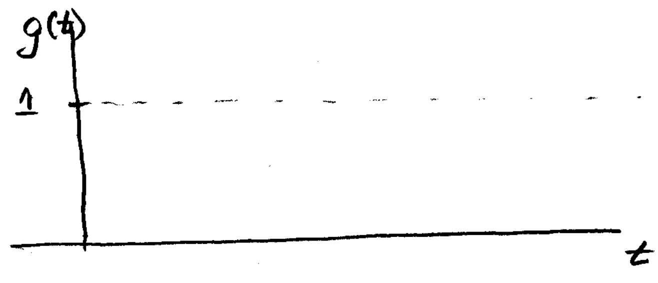
$$\frac{V_i - (1+sR_bC_b)V_o}{R_i} + \frac{V_o - (1+sR_bC_b)V_o}{1/sC_f} = sC_b V_o$$

- after tidying

$$V_o(s) = \frac{1}{s^2 R_bC_bR_iC_f + sC_b(R_b+R_i) + 1} \cdot V_i(s)$$

$$= \frac{1/(R_bC_bR_iC_f)}{s^2 + \frac{1}{R_{\parallel}C_f}s + \frac{1}{R_bC_bR_iC_f}} \cdot V_i(s)$$

- what type of response?
  - check DC gain $H(0) =$
  - so step response might look like



What freedom in choosing ω_c, ζ ?

- Design freedom in choice of ω_0, ζ? Can we make it underdamped? Resonant?

  - standard parameterization

    $$s^2 + 2\zeta\omega_0 s + \omega_0^2$$

  - so $\omega_0 = \dfrac{1}{\sqrt{R_b C_b R_i C_f}}$ $\qquad$ reasonable ranges of component values for audio

    $$= \frac{1}{R_i C_f \sqrt{\gamma\rho}} \qquad \gamma = C_b/C_f$$

    $$\rho = R_b/R_i$$

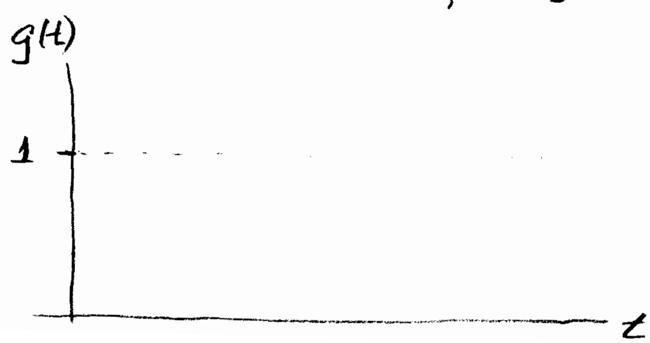
  - and $2\zeta\omega_0 = \dfrac{1}{R_{\|} C_f}$ $\qquad R_{\|} = \dfrac{R_b R_i}{R_b + R_i} = \dfrac{\rho}{1+\rho} R_i$

    $$\zeta = \frac{1+\rho}{2\rho R_i C_f} \cdot R_i C_f \sqrt{\gamma\rho} = \frac{1}{2}(1+\rho)\sqrt{\frac{\gamma}{\rho}}$$

  - Difficult to get very low ζ, for resonance, without large ratios of components.

  - But the Butterworth lowpass, $\zeta = \frac{1}{\sqrt{2}}$, is easy. One of many ways: $\rho = 1$, $\gamma = \frac{1}{2}$

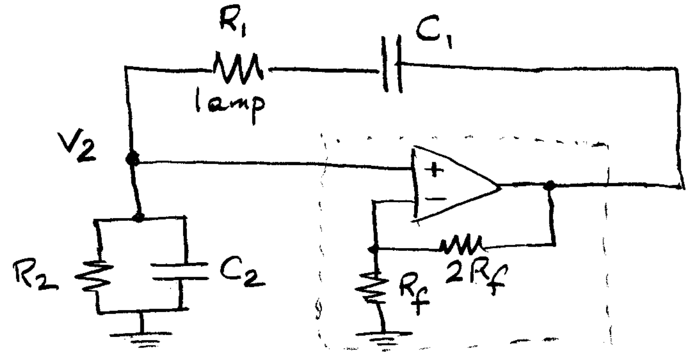
    so $R_b = R_i$, $C_b = 2C_f$



from p. 2.5.9

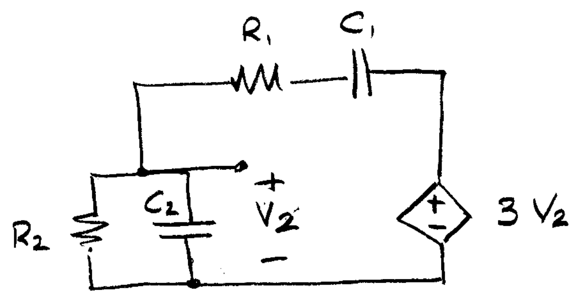
Wien Bridge Oscillator

- DC & L example 9.13 done with Laplace transform.



- We could work it out like a new problem. Instead note that the dotted line encloseds a non-inverting amp with voltage gain $(2R_f + R_f)/R_f = 3$.

Equivalent circuit:



- KCL at V_2:

$$\frac{3V_2 - V_2}{R_1 + \frac{1}{sC_1}} = V_2\left(sC_2 + \frac{1}{R_2}\right)$$

RHS uses admittance

- Rearranging gives

$$\left(s^2 + s\left(\frac{1}{R_2C_2} + \frac{1}{R_1C_1} - \frac{2}{R_1C_2}\right) + \frac{1}{R_1C_1R_2C_2}\right) V_2(s) = 0$$

a homogeneous equation.

- The negative sign suggests we can make ζ as small as we want.

- Design:
  - Simplify with $C_1 = C_2 = C$, so

$$\left(s^2 + s \cdot \frac{1}{C}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{1}{R_1R_2C^2}\right) V_2(s) = 0$$

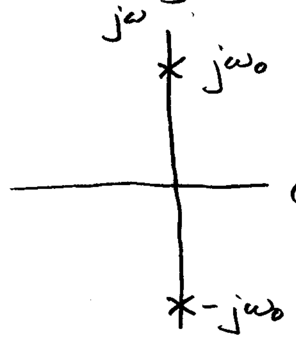
  - We have

$$\omega_0 = \frac{1}{C\sqrt{R_1R_2}}$$

$$2\zeta\omega_0 = \frac{1}{C}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

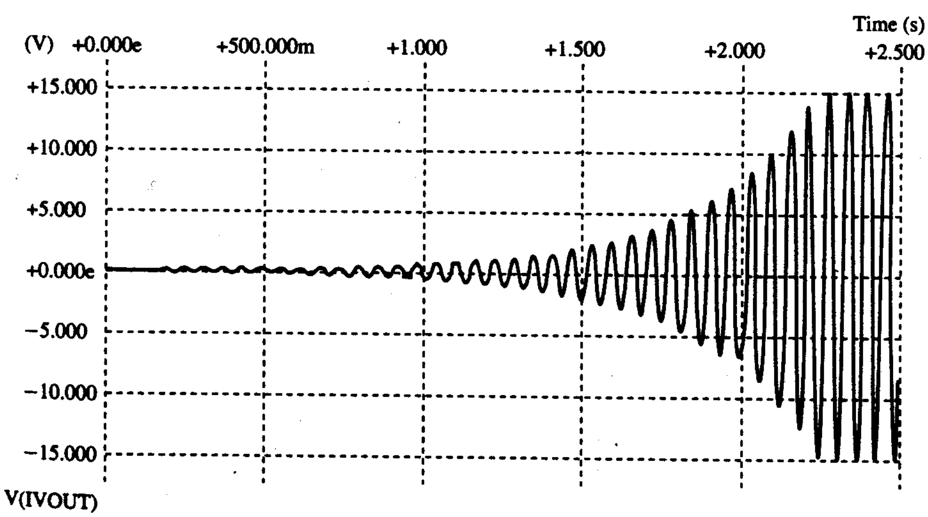
$$\zeta = \frac{1}{2}\left(\sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}}\right) = \begin{cases} >0, & R_1 > R_2 \\ 0, & R_1 = R_2 \\ <0, & R_1 < R_2 \end{cases}$$

- for sine wave solution, not growing or dying, we want



that is, $\zeta = 0$

- So choose $R_1 \lesssim R_2$. Oscillations grow, but increasing current through R_1 causes its resistance to grow (it's a lamp), so it stabilizes.
- Not a clean sine wave if it saturates the op amp



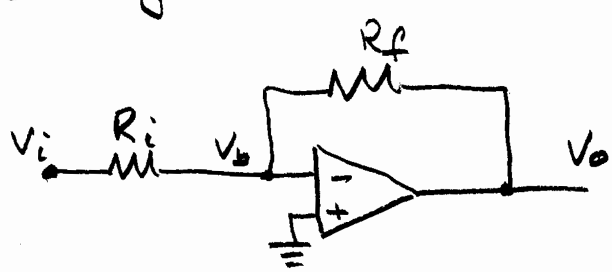
(b)

Figure 9.18 (a) Schematic diagram of Wien bridge oscillator. (b) Voltage response showing growing oscillation clamped at ±15 V due to saturation effects of op amp.

- If poles are in RHP, and the output grows, then the power in the output grows, too. Where does that energy come from?

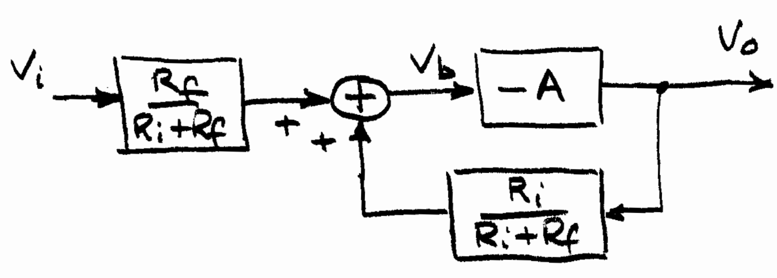
- Unstable systems generally have access to additional power and usually involve feedback – variation in the output causes more drive at the input in a way that further increases the variation.

- Benign feedback:



$$V_b = \frac{R_f}{R_i+R_f} V_i + \frac{R_i}{R_i+R_f} V_o$$

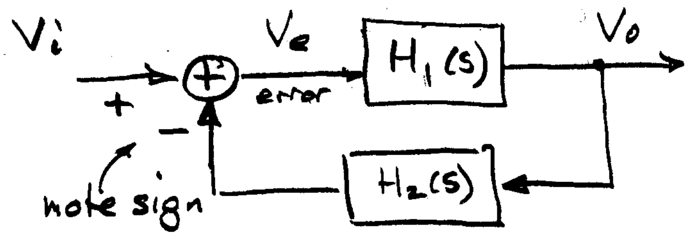
$$V_o = -A V_b$$



$$V_o = -A\left(\frac{R_f}{R_i+R_f} V_i + \frac{R_i}{R_i+R_f} V_o\right)$$

$$V_o = -\frac{R_f}{R_i} \frac{A}{A+1+\frac{R_f}{R_i}}$$

More commonly diagrammed as



$$V_o = H_1\left(V_i - H_2 V_o\right)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{H_1(s)}{1 + H_1(s)\,H_2(s)} = \frac{\text{"forward gain"}}{\text{"one plus the loop gain"}}$$

"forward gain"
over
"one plus the loop gain"

Appropriate choice of $H_2(s)$ can:

- stabilize an unstable $H_1(s)$
- make V_o track V_i in the face of changes in $H_1(s)$
- improve the speed of responce of $H_1(s)$

but, done wrongly, it can cause instability by moving poles to RHP