4 FREQUENCY RESPONSE

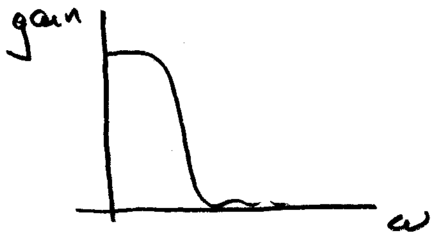
DC & L Chapter 15

- "Frequency response" refers to the steady-state response of the circuit to a sinusoidal input. It specifies the gain and phase shift relating output to input as a function of input frequency.

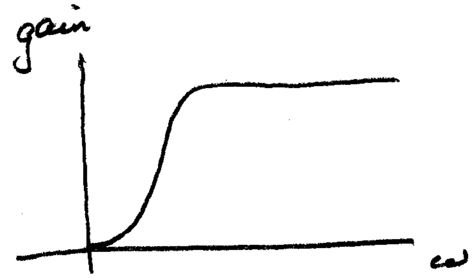
- Frequency response has this physical interpretation only for stable systems (poles in the LHP, or $\int_{0^-}^{\infty} |h(t)|\,dt < \infty$), for which the transients (contributions to the output from $H(s)$ poles) die out.

- The frequency response is as general a description of circuit behaviour as impulse response, step response, transfer function or DE, since any one can be obtained from any other.

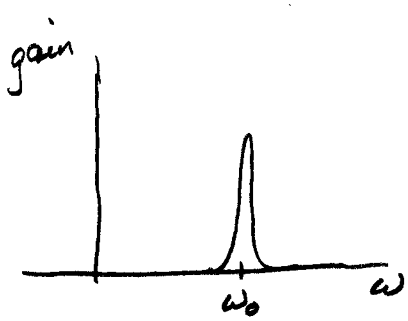
- We often characterise circuits by the gain portion of their frequency response:



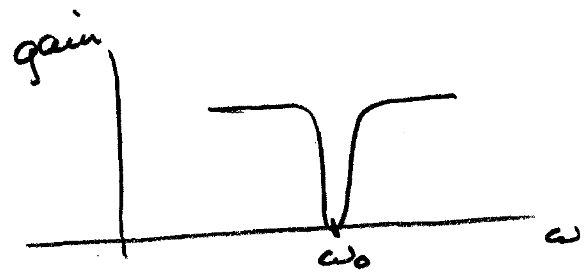
Lowpass: good for suppressing high freq. noise contaminating a low freq signal.



Highpass: good for removing unwanted DC offset; a change detector.



Narrow bandpass, tuned or resonant filter: good for extracting a periodic component in the input or for selecting one tone or channel.



Notch filter: good for removing unwanted tones or channels, or for detecting change from periodic input (with period $2\pi/\omega_0$)

but phase shift as a function of frequency is also important!

4.1 Frequency Response by Laplace Transform

- If $H(s)$ is the transfer function, $H(j\omega)$ is the frequency response: $|H(\omega)|$ is the amplitude gain, $\measuredangle H(j\omega)$ or $\arg[H(j\omega)]$ is the phase shift.

  A demonstration of this below.

- Consider the input

$$x(t) = \cos(\omega t)\, u(t) \longleftrightarrow X(s) = \frac{s}{s^2+\omega^2}$$

for some frequency ω. The output is

$$Y(s) = H(s) \frac{s}{s^2+\omega^2}$$

$$y(t) = \sum_i \mathrm{Res}_i(t) \qquad \text{residues of } H(s)\frac{s e^{st}}{s^2+\omega^2}$$

poles: $\pm j\omega$ plus those of $H(s)$

- By hypothesis, poles of $H(s)$ contribute transient terms, so steady state output

$$y(t) = 2\,\mathrm{Re}\left[H(s)\frac{s\, e^{st}}{s+j\omega}\bigg|_{s=j\omega} \right] = 2\,\mathrm{Re}\left[H(j\omega)\frac{j\omega}{2j\omega} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[H(j\omega)\, e^{j\omega t} \right]$$

– As usual, we can write this in rectangular or polar.

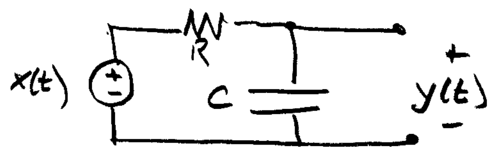
$$y(t) = |H(j\omega)| \, \mathrm{Re}\left[e^{j \measuredangle H(j\omega)} e^{j\omega t} \right]$$

$$= |H(j\omega)| \cos(\omega t + \measuredangle H(j\omega)) \qquad \text{usual}$$

or with $H(j\omega) = H_r(\omega) + j\,H_i(\omega)$,

$$y(t) = H_r(\omega)\cos(\omega t) - H_i(\omega)\sin(\omega t) \qquad \text{less common}$$

• example



$$H(s) = \frac{1/sC}{R + \frac{1}{sC}} = \frac{1/RC}{s + \frac{1}{RC}} = \frac{1/\tau}{s + 1/\tau}$$

$$y(t) = \mathcal{L}^{-1}\left[H(s) \frac{s}{s^2+\omega^2} \right] = \mathcal{L}^{-1}\left[\frac{s/\tau}{(s + 1/\tau)(s^2+\omega^2)} \right]$$

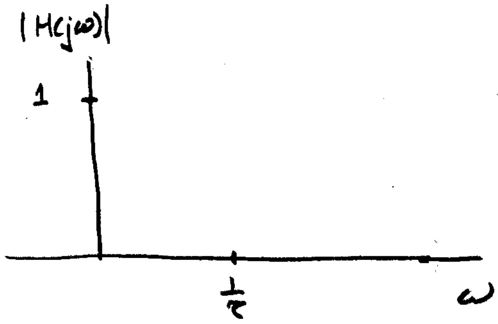
$$= \mathrm{Re}\left[H(j\omega) e^{j\omega t} \right] - \underbrace{\frac{1}{1+\tau^2\omega^2} e^{-t/\tau}}_{\text{dies away}}$$

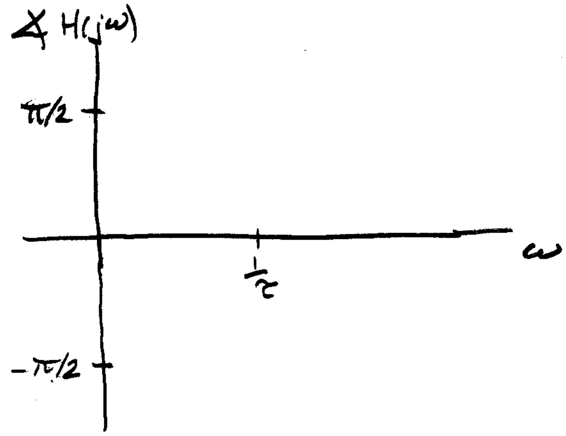
in steady state:

$$= \mathrm{Re}\left[\frac{1/\tau}{j\omega + 1/\tau} \cdot e^{j\omega t} \right]$$

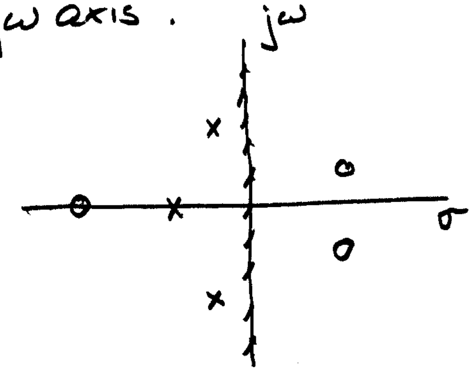
$$= \frac{1}{\sqrt{1+\omega^2\tau^2}} \cos(\omega t - \tan^{-1}(\omega\tau))$$

Its frequency response is $H(j\omega) = \dfrac{1/\tau}{j\omega + \frac{1}{\tau}}$ or, equivalently, the two real functions, like $|H(j\omega)|$ and $\measuredangle H(j\omega)$





- Interpretation:
  - Frequency response $H(j\omega)$ is $H(s)$ along the $j\omega$ axis.



The $j\omega$ axis is in the ROC, since all poles are in LHP

  - Frequency response is Fourier transform of $h(t)$.

$$H(j\omega) = H(s)\big|_{s=j\omega} = \int_{0^-}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

  - Recover $H(s)$ from freq response with $\omega \leftarrow \frac{s}{j} = -js$

- More examples — elementary circuits.

  - differentiator

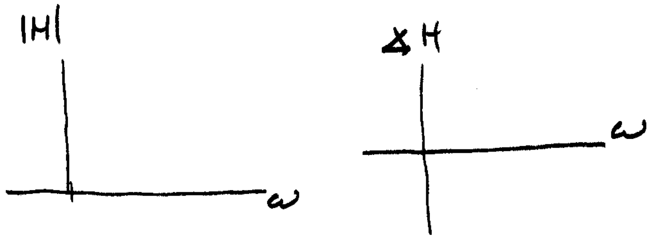
    $y(t) = \frac{d}{dt} x(t)$, for $x(t) = \cos(\omega t)$, $y(t) = -\omega \sin(\omega t)$

    $Y(s) = s\, X(s)$ so $H(s) = s$

    Then frequency response is $H(j\omega) = j\omega$. We have

    $|H(j\omega)| = \omega$ $\quad \measuredangle H(j\omega) = \pi/2$

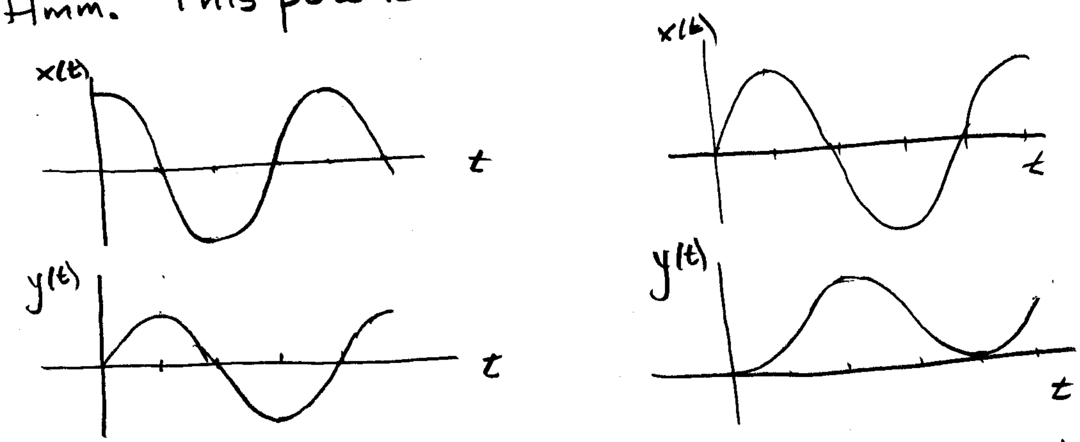
    Do these values make sense?



  - integrator, $y(t) = \int_{0^-}^{t} x(\alpha)\, d\alpha$; if $x(t) = \cos(\omega t)$, $y(t) = \frac{\sin(\omega t)}{\omega}$

    $Y(s) = \frac{1}{s} X(s)$ so $H(s) = \frac{1}{s}$

    Hmm. This pole is not in the LHP.

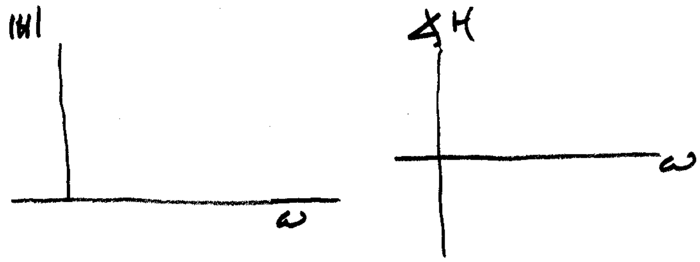


a persistent constant offset from pole at $s=0$

Sometimes we're willing to overlook the non-transient "transient" and loosely refer to the particular solution's frequency response

$$H(j\omega) = \frac{1}{j\omega}$$

Does it make sense?



– Delay device.

$$y(t) = x(t-T) \qquad x(t) = \cos(\omega t),$$

$$y(t) = \cos(\omega(t-T))$$

$$= \cos(\omega t - \omega T)$$

We have $H(s) = e^{-sT}$ (not realizable by finite order)

so freq response $H(j\omega) = e^{-j\omega T}$.

Sensible?