4.4 Resonance Phenomena

DC+L Chapt 17

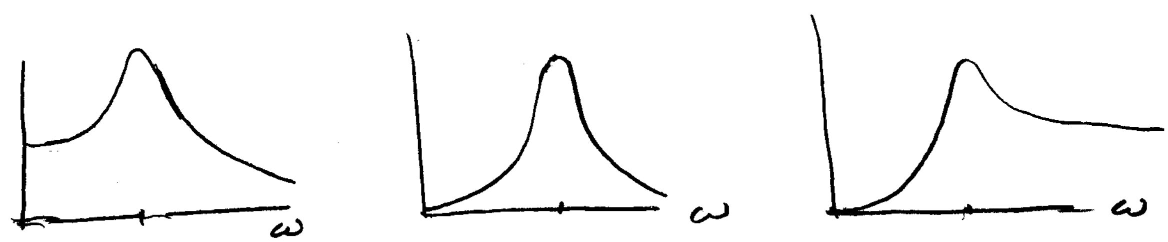
- We're going to look at lightly damped second order circuits in a little more detail, pulling together the various threads.

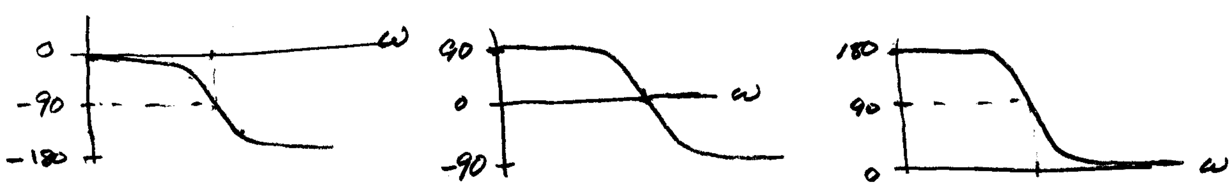
4.4.1 Standard Forms

- It's convenient to adopt these three standard forms:

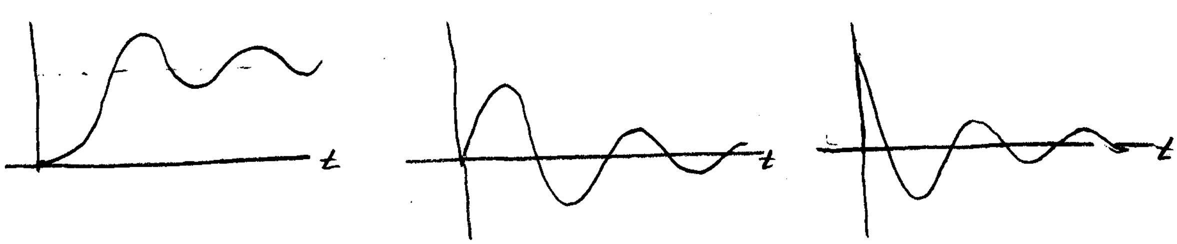
type:	lowpass	bandpass	high pass
form:	$\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$	$\frac{2\zeta\omega_0 s}{s^2+2\zeta\omega_0 s+\omega_0^2}$	$\frac{s^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$
low freq asympt:	1 (0dB)	+20 dB/dec	+40 dB/dec
hi freq asympt:	−40 dB/dec	−20 dB/dec	1 (0dB)

freq resp:

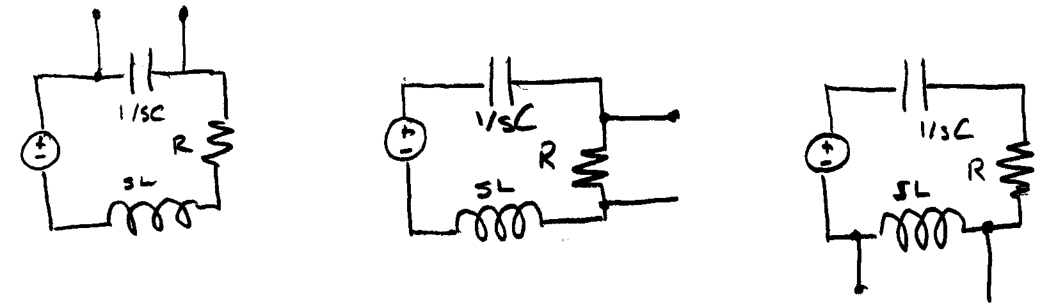




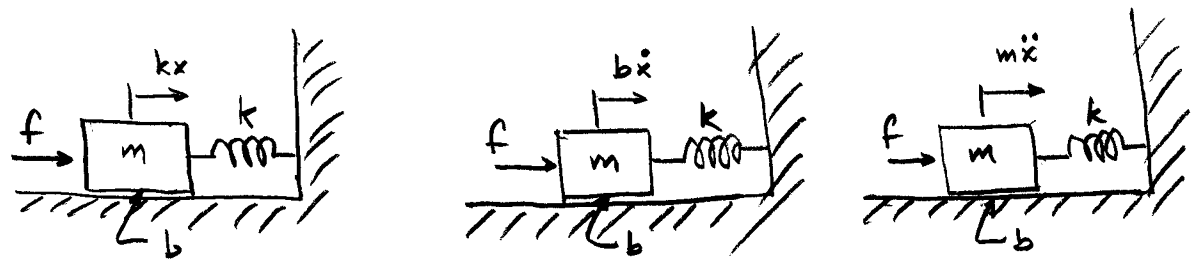
step resp:



typical circuit:

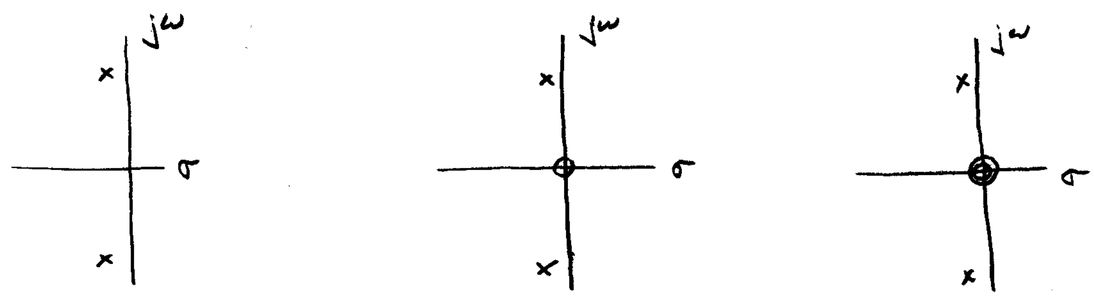


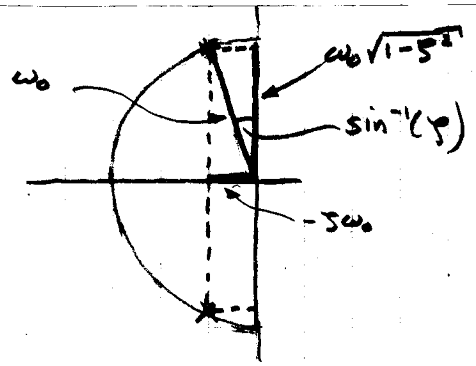
$$\frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad \frac{s\,R/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



spring force | friction force | accel. force

$$\frac{k/m}{s^2 + \frac{b}{m}s + \frac{k}{m}} \qquad \frac{s\,b/m}{s^2 + \frac{b}{m}s + \frac{k}{m}} \qquad \frac{s^2}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$





pole locations $\omega_0\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)$, $\zeta < 1$

as ζ decreases from 1 to 0, poles move from $-\omega_0$ on real axis to $\pm j\omega_0$ on imaginary axis along a circle of radius ω_0 (natural frequency)

4.4.2 Frequency Response and Q

- Look at the frequency response of lightly damped systems. Easiest to analyze the bandpass version:

$$H(s) = \frac{2\zeta\omega_0 s}{s^2+2\zeta\omega_0 s+\omega_0^2} = \frac{1}{\frac{s}{2\zeta\omega_0}+1+\frac{\omega_0}{2\zeta s}}$$

Substitute $s = j\omega$ to obtain frequency response

$$H(j\omega) = \frac{1}{\frac{j\omega}{2\zeta\omega_0}+1+\frac{\omega_0}{j2\zeta\omega}} = \frac{1}{1+j\left(\frac{u}{2\zeta}-\frac{1}{2\zeta u}\right)}$$

$u = \omega/\omega_0$

$\omega = \omega_0 u$

$$= \frac{1}{1+jQ\left(u-\frac{1}{u}\right)}$$ where $Q = \frac{1}{2\zeta}$

Note the geometric symmetry about ω_0:

$$H(j\omega_0 u) = H^*(j\omega_0/u)$$

This means symmetry on a log frequency scale:

→ if $u = \frac{1}{2}$, $\log u = -0.3$

→ if $u = 1$, $\log u = 0$

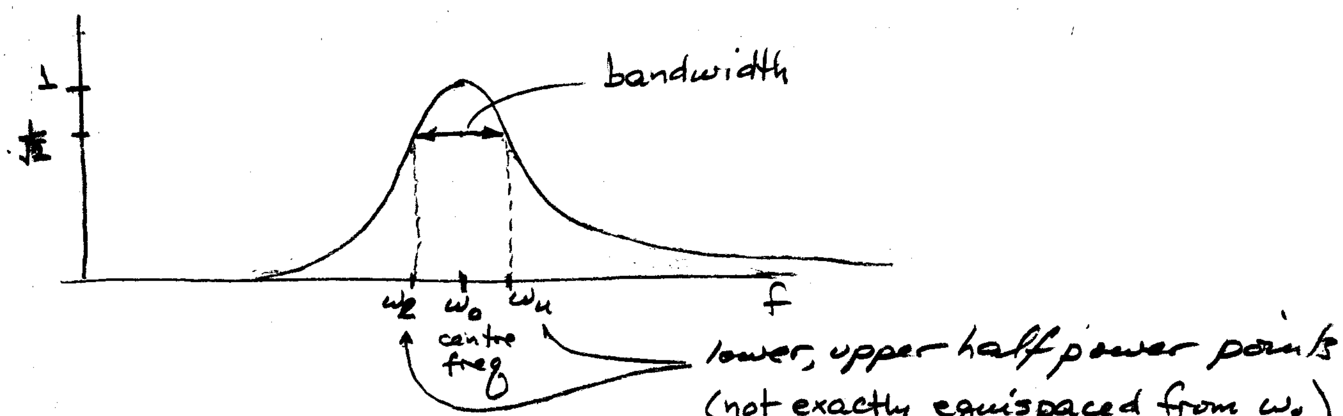
→ if $u = 2$, $\log u = 0.3$

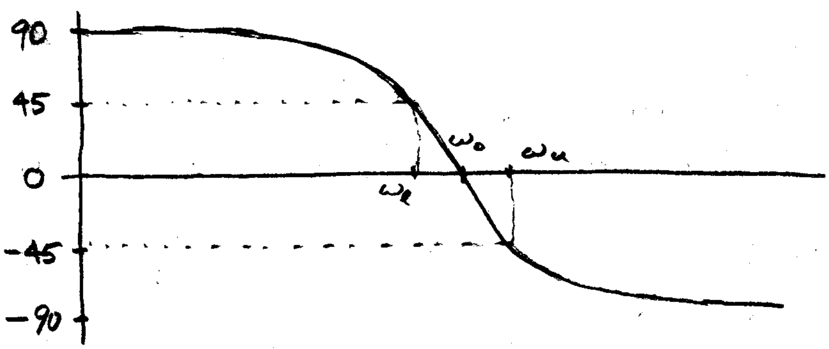
$|H|$ is the same

$\measuredangle$ is negated

- We can easily pick off several points on the frequency response

ω	$\lvert H(j\omega)\rvert$	$\arg[H(j\omega)]$	comment
0	0	$90°$	
ω_l	$\frac{1}{\sqrt{2}}$ (-3dB)	$45°$	lower half pwr pt $\left(u - \frac{1}{u} = -\frac{1}{Q}\right)$
ω_0	1 (0dB)	$0°$	resonant max
ω_u	$\frac{1}{\sqrt{2}}$ (-3dB)	$-45°$	upper half pwr pt $\left(u - \frac{1}{u} = \frac{1}{Q}\right)$
∞	0	$-90°$	





- ω_0 is the frequency of maximum resonance ω_m for this bandpass form; small shift down for lowpass, up for highpass

- From $H(j\omega) = \dfrac{1}{1+jQ(u-\frac{1}{u})}$ we see that,

as Q increases, the half power points move closer to ω_o, so the bandwidth decreases. The rapid changes in amplitude and phase take place over a smaller range of frequencies.

- Location of half power points?
  - Exact calculation requires solving a quadratic for ω_ℓ $\left(u_\ell - \frac{1}{u_\ell} = -\frac{1}{Q}\right)$ and a quadratic for ω_u $\left(u_u - \frac{1}{u_u} = \frac{1}{Q}\right)$.
  - A simple approximation of $H(j\omega)$ is good for high Q and/or ω near ω_o:

    define $\epsilon = \frac{\omega}{\omega_o} - 1 = u - 1$

    so $\frac{1}{u} = \frac{1}{1+\epsilon} = 1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} + \dots,$

    and $u - \frac{1}{u} \approx (1+\epsilon) - (1-\epsilon) = 2\epsilon$

    giving $H(j\omega) \approx \dfrac{1}{1+j2Q\epsilon}$
  - Thus upper and lower half power points are at $\epsilon = \pm 1/2Q$ or $\omega_{\{u \atop \ell\}} = \omega_o \pm \frac{\omega_o}{2Q}$ (approximately)

– The bandwidth is

$$\beta = \omega_u - \omega_\ell = \frac{\omega_0}{Q} \quad *$$

so another common interpretation of Q is the reciprocal fractional bandwidth

$$Q = \frac{\omega_0}{\beta}$$ doesn't matter if Hz or rad/sec

Usually, $Q \approx \omega_m/\beta$, but $\omega_m = \omega_0$ for bandpass.

Also,

$$\beta = \frac{\omega_0}{Q} = 2\zeta\omega_0$$

so standard form is

$$H_{bp}(s) = \frac{\beta s}{s^2 + \beta s + \omega_0^2}$$

example

$$H(s) = \frac{2s}{s^2 + 2s + 400}$$

It looks underdamped. What are:

- the bandwidth?
- the natural frequency?
- the Q?
- the damping factor?

* This is exact, despite the ω_ℓ, ω_u approximations on p. 4.4.5. From $u_\ell - \frac{1}{u_\ell} = -\frac{1}{Q}$, we have $u_\ell = -\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}$, and from $u_u - \frac{1}{u_u} = \frac{1}{Q}$, we have $u_u = \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}$. So $u_u - u_\ell = \frac{1}{Q}$ and $\beta = \omega_u - \omega_\ell = \frac{\omega_0}{Q}$

- Finally, recall that we started with the bandpass form, and found $\omega_m = \omega_o$ (frequency of max response) Alternatives:
  - The lowpass form $H_{lp}(s) = \dfrac{\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}$

    lacks ω in the numerator, so ω_m shifts slightly down:
    $$\omega_m = \omega_o\sqrt{1-2\zeta^2} \approx \omega_o(1-\zeta^2) = \omega_o\left(1 - \frac{1}{4Q^2}\right)$$
    and the phase runs from $0°$ down to $-180°$.

  - The highpass form $H_{hp}(s) = \dfrac{s^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}$

    has ω^2 in the numerator, ω_m shifts slightly up:
    $$\omega_m = \frac{\omega_o}{\sqrt{1-2\zeta^2}} \approx \omega_o(1+\zeta^2) = \omega_o\left(1 + \frac{1}{4Q^2}\right)$$
    and the phase runs from $+180°$ down to $0°$.

  - Both shifts are small compared with distance to half pwr points. From p 4.4.5,
    $$\omega_u - \omega_o \approx \frac{\omega_o}{2Q}$$
    Compare with shift of ω_m: $\dfrac{\omega_o}{4Q^2}$

4.4.3 Time Response and Q

- Even though resonance is usually described in the frequency domain, we must also examine the behaviour of resonant circuits in the time domain. Why?

  - Requirements are sometimes expressed in time (*e.g.*, response time)

  - Energy and power variations are most easily discussed in the time domain.

- Again, we'll use the bandpass form to illustrate the points.

$$H(s) = \frac{2\cdot\zeta\cdot\omega_o\cdot s}{s^2 + 2\cdot\zeta\cdot\omega_o\cdot s + \omega_o^2}$$

$$\zeta = \frac{1}{2\cdot Q}$$

$$= \frac{\frac{\omega_o}{Q}\cdot s}{s^2 + \frac{\omega_o}{Q}\cdot s + \omega_o^2}$$

$$\beta = \frac{\omega_o}{Q}$$

$$= \frac{\beta\cdot s}{s^2 + \beta\cdot s + \omega_o^2}$$

- For the bandpass form, we have the step response

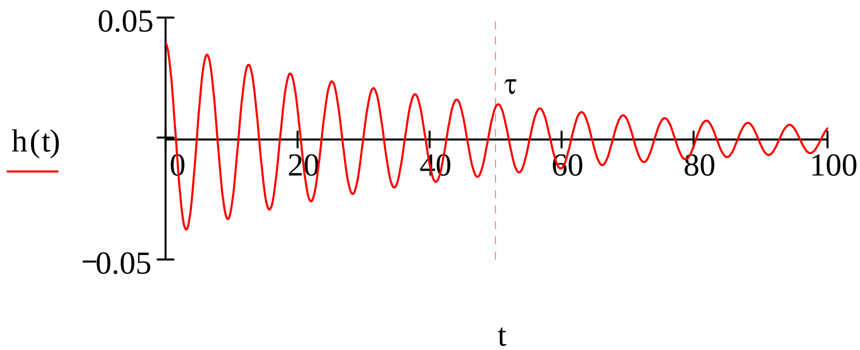
$$g(t) = \frac{2\cdot\zeta}{\sqrt{1-\zeta^2}}\cdot e^{-\zeta\cdot\omega_o\cdot t}\cdot\sin\left(\sqrt{1-\zeta^2}\cdot\omega_o\cdot t\right) \qquad t \ge 0$$

$$= \frac{2\cdot\zeta}{\sqrt{1-\zeta^2}}\cdot e^{\frac{-t}{\tau}}\cdot\sin(\omega_d\cdot t) \qquad t \ge 0$$

so we have $\tau = \frac{1}{\zeta\cdot\omega_o} = \frac{2\cdot Q}{\omega_o} = \frac{2}{\beta}$ time constant related to bandwidth and Q (it's the reciprocal of the half bandwidth)

and the impulse response

$$h(t) = 2\cdot\zeta\cdot\omega_o\cdot e^{-\zeta\cdot\omega_o\cdot t}\cdot\left(\cos(\omega_d\cdot t) - \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\sin(\omega_d\cdot t)\right) \qquad t \ge 0$$



$\omega_o = 1$

$\zeta = 0.02$

$\tau = 50$

$Q = 25$

$\beta = 0.04$

cycles per time constant: $\frac{\omega_o\cdot\sqrt{1-\zeta^2}}{2\cdot\pi}\cdot\frac{1}{\zeta\cdot\omega_o} = \frac{1}{2\cdot\pi\cdot\zeta} = \frac{Q}{\pi}$

approx

- Now look at how quickly this resonant circuit responds to a input tone at the resonant frequency. Solve by transforms:

input

$$x(t) = \sin(\omega_o \cdot t)$$

input transform

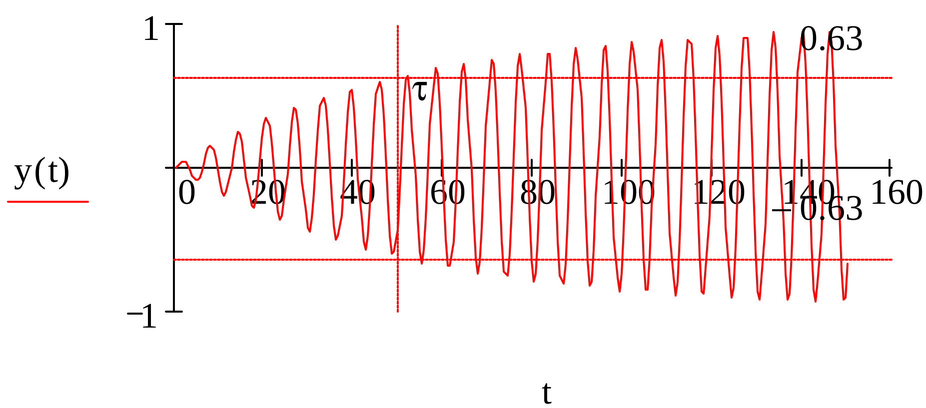
$$X(s) = \frac{\omega_o}{s^2 + \omega_o^2}$$

$$Y(s) = \frac{2 \cdot \zeta \cdot \omega_o \cdot s}{s^2 + 2 \cdot \zeta \cdot \omega_o \cdot s + \omega_o^2} \cdot \frac{\omega_o}{s^2 + \omega_o^2}$$

output transform

$$y(t) := \left(\frac{-e^{-\zeta \cdot \omega_o \cdot t}}{\sqrt{1 - \zeta^2}} \cdot \sin(\omega_d \cdot t) + \sin(\omega_o \cdot t) \right)$$

output (after some work)



$\omega_o = 1$

$\zeta = 0.02$

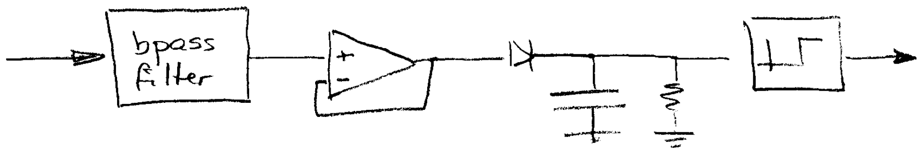
$\tau = 50$

$Q = 25$

$\beta = 0.04$

There is an exponential rise in amplitude, much like that of a first order lowpass responding to a unit step. Not too surprising, if you check out the pole-zero diagrams of $Y(s)$ for the lowpass and bandpass cases.

Example: tone detection



A tone-coded system for remote control of a radio relay is arranged so that, when the relay receives a 1 kHz tone, it turns on its main transmitter. The sketch shows a possible design.

(a) In order to reject as much noise and interference as possible, you want the filter to be as selective as possible. Yet one of the specifications is that the transmitter should turn on within 5 ms of the onset of the tone. What is the maximum Q you can use?

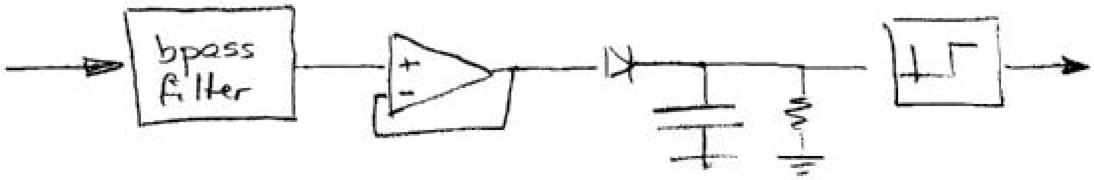
(b) Having designed the detector for one tone command, you decide to add another tone at a higher frequency to control the signal source to the transmitter. The frequency of the second tone now has to be determined. You decide that the response of the first filter to this second tone should be 15 dB less than its response to its own tone, to ensure there are no false triggers. How closely spaced can the tones be?

(c) If we relaxed the response time requirement, could we space the tones more closely?

(d) What if we don't know the tone amplitude very precisely? Any solutions?

Solution:

This circuit should do it for us:



(a) The response of a second order bandpass filter to a tone at its resonant frequency is a sinusoid at the same frequency, with amplitude increasing from zero to its maximum value with some time constant. The envelope is much like that of a first order lowpass circuit.

Set the threshold at 63% of the maximum output of the tuned circuit. This corresponds to one time constant in the rise of the envelope (*i.e.*, it is $1\text{-}e^{-1}$). The time constant of a 2nd order bandpass is $\tau = 1/\zeta\omega_o$ and the Q of such a circuit is $1/2\zeta$. Combining them with the requirement of a maximum 5 ms time constant, we have

$$\omega_o := 2\cdot\pi\cdot 1000\cdot\mathrm{Hz} \qquad \tau_{max} := 5\cdot 10^{-3}\cdot\mathrm{sec}$$

$$Q_{max} := \frac{\omega_o\cdot\tau_{max}}{2} \qquad Q_{max} = 15.708$$

Related parameters:

$$\zeta_{min} := \frac{1}{2\cdot Q_{max}} \qquad \zeta_{min} = 0.032$$

min damping factor

$$\omega_d := \omega_o \cdot \sqrt{1 - \zeta_{min}^{\ 2}} \qquad \omega_d = 6.28 \times 10^3 \, s^{-1}$$

damped natural frequency (rad/s)

$$\beta_{min} := \frac{\omega_o}{Q_{max}} \qquad \beta_{min} = 400 \, s^{-1}$$

bandwidth (rad/s)

If you forget some of these relationships, you can work it out like this: the transform of the output is the product of the transform of the sine wave input and the transfer function:

$$V_o(s) = \frac{\omega_o}{s^2 + \omega_o^{\ 2}} \cdot \frac{\beta \cdot s}{s^2 + \beta \cdot s + \omega_o^{\ 2}}$$

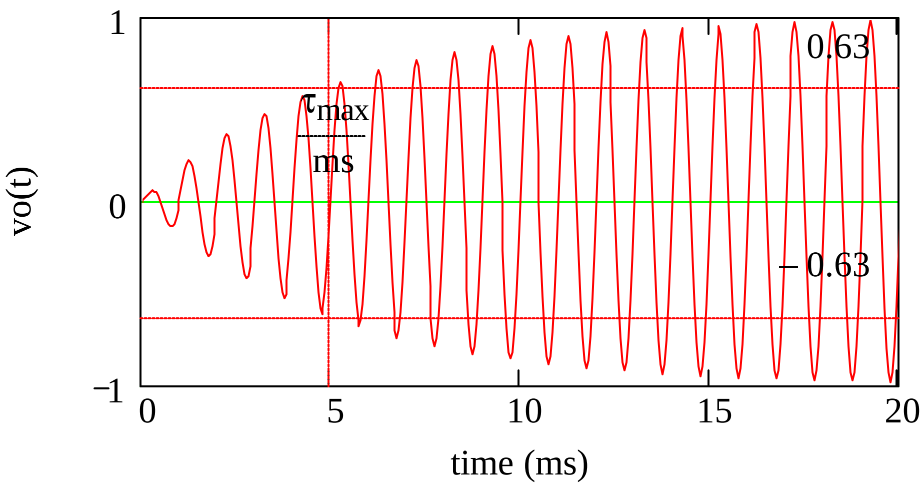
where the bandwidth $\beta = 2 \cdot \zeta \cdot \omega_o = \frac{\omega_o}{Q} = \frac{2}{\tau}$

Inversion yields

$$v_o(t, \omega_o, \beta, \omega_d) := \sin(\omega_o \cdot t) - \frac{\omega_o}{\omega_d} \cdot e^{\frac{-\beta \cdot t}{2}} \cdot \sin(\omega_d \cdot t)$$

where the damped natural frequency is $\omega_d = \omega_o \cdot \sqrt{1 - \zeta^2}$

Now plot it, using the values determined by the maximum time constant.



(b) We now have the *Q* and the natural frequency, and therefore the frequency response of the bandpass. In locating the second tone, we need to find the frequency at which the response is 15 dB down from its value when excited by the first tone. The transfer function is

$$H(\omega) := \frac{1}{1 + j\cdot Q_{max}\cdot\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}$$

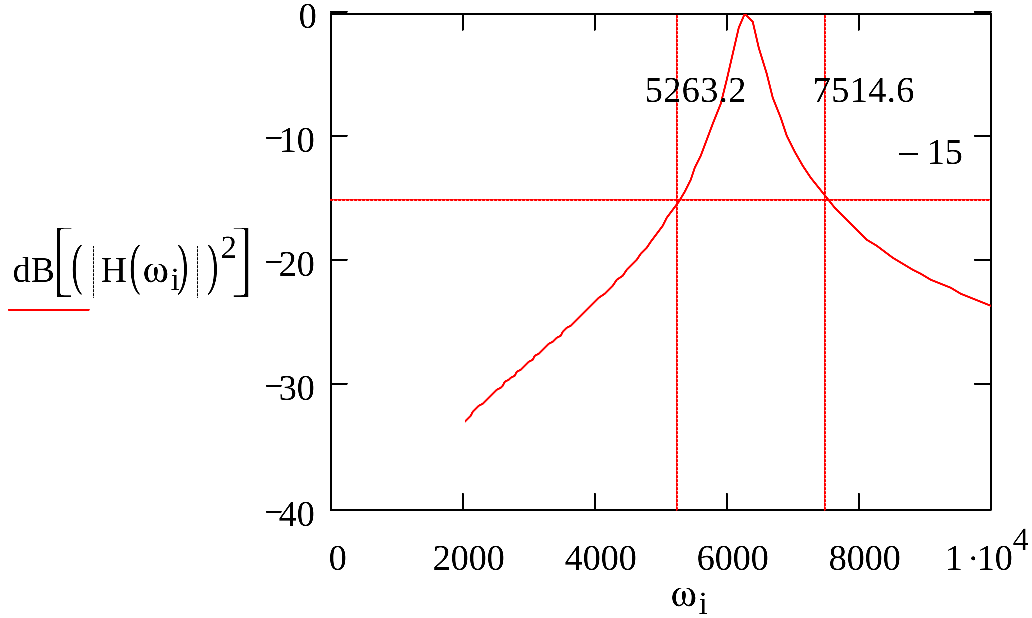
Now define dB and plot the magnitude of the response in dB

$$dB(x) := \text{if}\left(x < 10^{-16}, -160, 10\cdot\log(x)\right)$$

$N_p := 100$ number of points (less 1) $i := 0 .. N_p$

plot using geometrically spaced frequencies (equispaced in log)

$\omega_{lo} := 2000$ $\omega_{hi} := 10000$ $r := \sqrt[N_p]{\frac{\omega_{hi}}{\omega_{lo}}}$ $\omega_i := \omega_{lo}\cdot r^i$



Graphically, we find that lower and upper -15 dB frequencies are 5263 rad/s and 7515 rad/s. Doing it analytically is messier. First note that

$$\left(\left|H(\omega)\right|\right)^2 = \frac{1}{1 + Q_{max}^2 \cdot \left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2}$$

and that 15 dB is $10^{0.1 \cdot 15} = 31.623$

Therefore $1 + Q_{max}^2 \cdot \left(u - \frac{1}{u}\right)^2 = 31.6$ where $u=\omega/\omega_o$.

Next, define $v=u^2$, so that

$$1 + \frac{Q_{max}^2}{v} \cdot (v - 1)^2 = 31.6$$

Expanding and collecting terms gives the quadratic

$$Q_{max}^2 \cdot v^2 + \left(-30.6 - 2 \cdot Q_{max}^2\right) \cdot v + Q_{max}^2 = 0$$

Substitute for Q_{max} $\quad 246.7 \cdot v^2 - 524 \cdot v + 246.7 = 0$

and solve to obtain the analytical counterpart of the graphical result:

$$v_{lo} := 0.7044 \qquad v_{hi} := 1.42$$

These give the lower and upper -15 dB points as the positive roots

$$\omega_{lo} := \omega_o \cdot \sqrt{v_{lo}} \qquad \omega_{hi} := \omega_o \cdot \sqrt{v_{hi}}$$

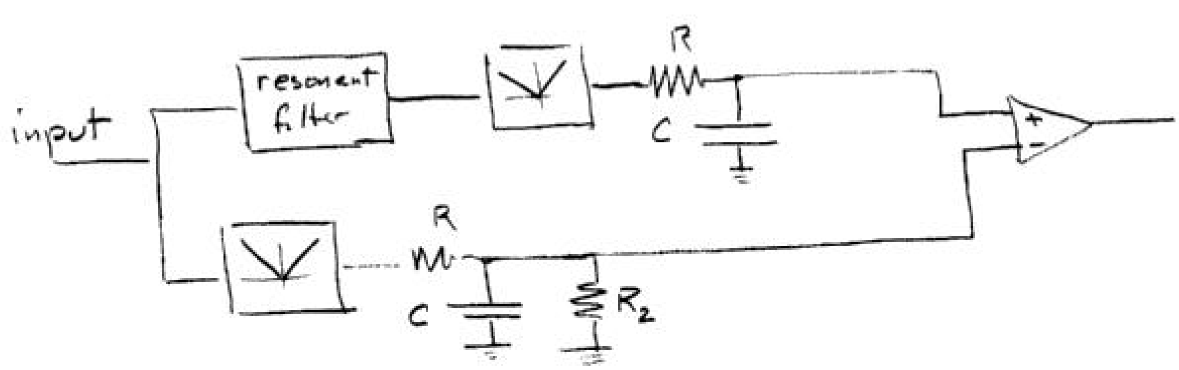
(the negative square roots give corresponding points on the image at negative frequencies).

$$\omega_{lo} = 5.273 \times 10^3 \frac{\text{rad}}{\text{sec}} \qquad f_{lo} := \frac{\omega_{lo}}{2 \cdot \pi} \qquad f_{lo} = 839.285\,\text{Hz}$$

$$\omega_{hi} = 7.487 \times 10^3 \frac{\text{rad}}{\text{sec}} \qquad f_{hi} := \frac{\omega_{hi}}{2 \cdot \pi} \qquad f_{hi} = 1.192 \times 10^3\,\text{Hz}$$

(c) If we relaxed the response time requirement and allowed the time constant to exceed 5 ms, then we could use a higher *Q* (assuming we could implement it). The higher *Q* would narrow the bandwidth and allow the higher tone to move closer to the one at 1 kHz and still meet the -15 dB requirement. Effectively, we have a longer memory with the increased time constant, and therefore a longer effective observation time. This allows greater frequency resolution; that is, greater ability to distinguish between nearby frequencies.

(d) The input to the threshold device is proportional to the amplitude of the incoming tone. If we don't know that amplitude, it's hard to set an absolute threshold. But how about a relative threshold? Measure the incoming signal amplitude, whether it's a tone at some frequency or something else altogether, and compare the amplitude of the bandpass filter output to it.



The voltage divider ratio $R_2/(R_2+R)$ sets the threshold on the bandpass output relative to the input amplitude.