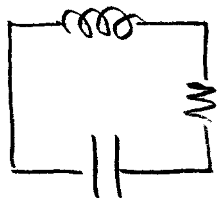
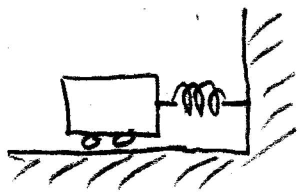
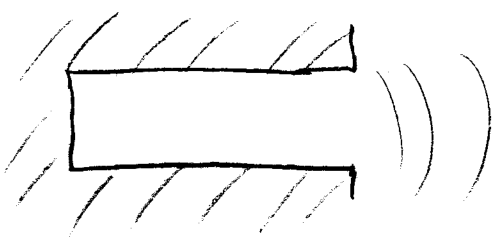
4.4.4 Physical Aspects of Resonance

- In all this bashing of equations, we must not lose sight of what is going on in the circuit.

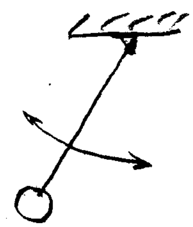
- Most resonance effects involve the interplay of two (or more) forms of energy storage.



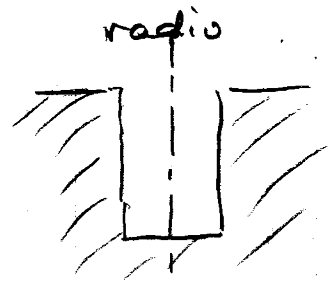




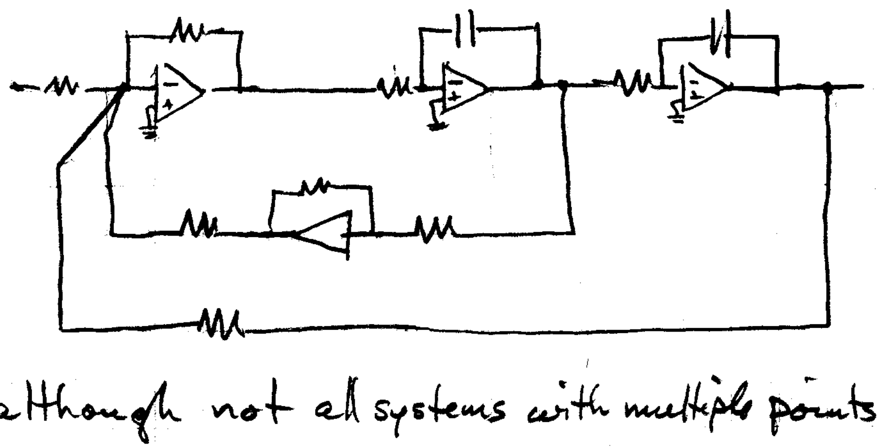
acoustic tube



pendulum

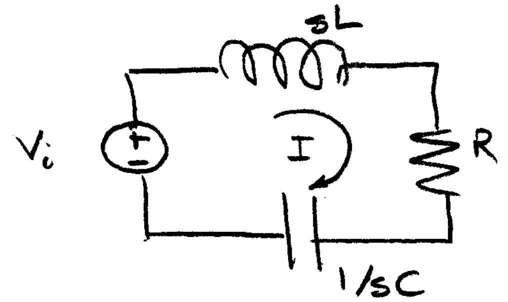


cavity resonator



although not all systems with multiple points of energy storage resonate.

- Consider series RLC for more detailed examination.



Kick it with a unit impulse ($V_i(s) = 1$) and obtain all three component voltages

cap
$$V_c(s) = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$v_c(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_d t), \quad t \geq 0, \quad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

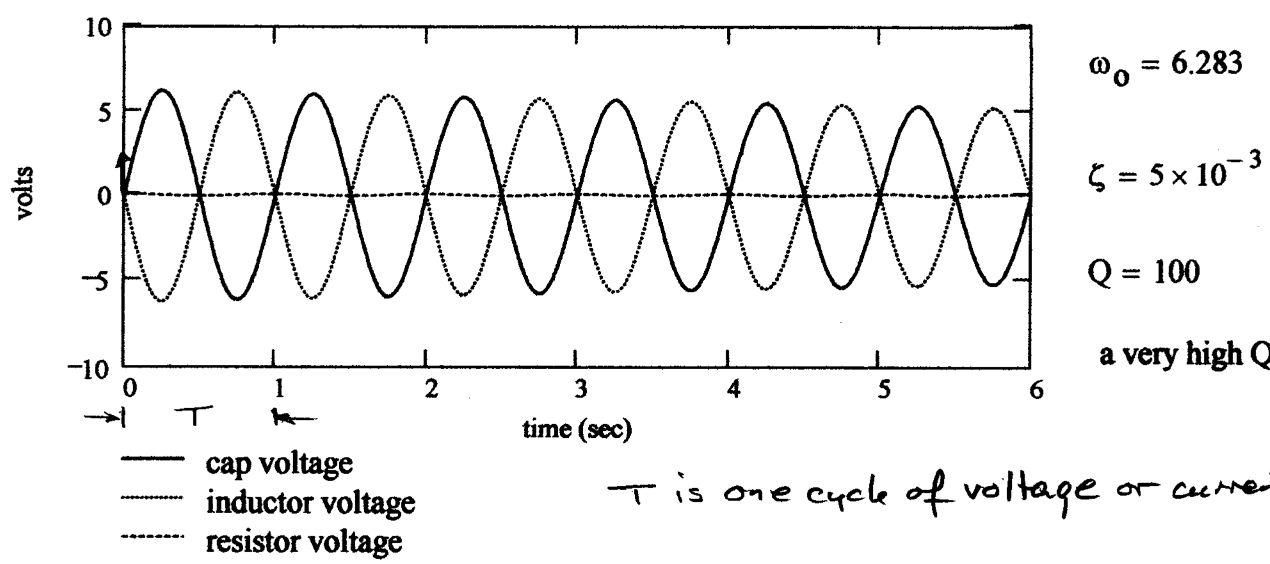
resistor
$$V_R(s) = \frac{s\,R/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$v_R(t) = 2\zeta\omega_0\, e^{-\zeta\omega_0 t}\left(\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t)\right)$$

inductor
$$V_L(s) = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$v_L(t) = \delta(t) - \omega_0 e^{-\zeta\omega_0 t}\left(2\zeta\cos(\omega_d t) + \frac{1-2\zeta^2}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

- Observations:
  - The impulse input causes a step change in current (see v_R, v_L).
  - The amplitudes of the cap and inductor voltages are larger than that of the resistor by a factor of about $\frac{1}{2\zeta} = Q$.
  - The inductor voltage $\left(\approx -\omega_0 e^{-\zeta\omega_0 t}\sin(\omega_d t)\right)$ leads the cap voltage $\left(\approx \omega_0 e^{-\zeta\omega_0 t}\sin(\omega_d t)\right)$ by about 180°.
  - So two large voltages, roughly equal and antiphase; they don't quite cancel; the difference is the resistor voltage.



T is one cycle of voltage or current

- Now look at the instantaneous energies:

$$E_c = \frac{1}{2} C v_c^2(t) \approx \frac{1}{2} C \omega_o^2 e^{-2\zeta\omega_o t} \sin^2(\omega_d t)$$

$$= \frac{1}{2L} e^{-2\zeta\omega_o t} \sin^2(\omega_d t)$$

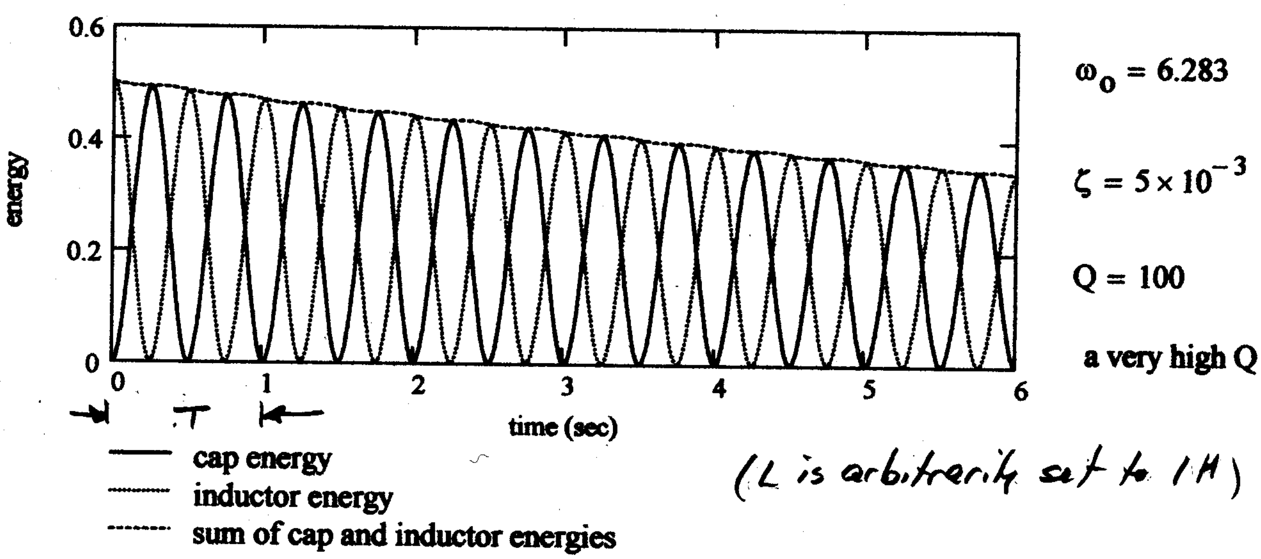
and

$$E_L = \frac{1}{2} L i^2(t) = \frac{1}{2} L \left(\frac{v_R(t)}{R}\right)^2$$

$$\approx \frac{1}{2} L \left(\frac{2\zeta\omega_o}{R} e^{-\zeta\omega_o t} \cos(\omega_d t)\right)^2$$

$$= \frac{1}{2L} e^{-2\zeta\omega_o t} \cos^2(\omega_d t)^2$$

Cap and inductor energies vary as $\sin^2(\omega_d t)$, $\cos^2(\omega_d t)$; energy sloshes back and forth:



(L is arbitrarily set to 1 H)

T is one cycle of voltage or current, but two cycles of energy.

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_o\sqrt{1-\zeta^2}}$$

- During T, the decay of energy is

$$e^{-2\zeta\omega_0 T} = \exp\left(-\frac{4\pi\zeta}{\sqrt{1-\zeta^2}}\right) \approx e^{-4\pi\zeta} = e^{-2\pi/Q}$$

and for small ζ (high Q) $e^{-2\pi/Q} \approx 1 - \frac{2\pi}{Q}$

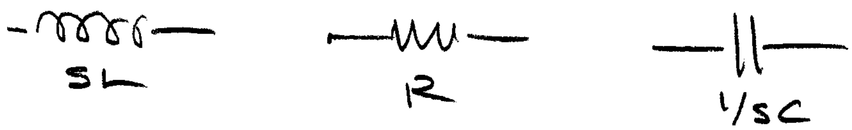
Therefore, the fractional energy loss per cycle is $2\pi/Q$.

e.g. for $Q = 100$, the system loses 0.0628, or 6.28% of its energy per cycle

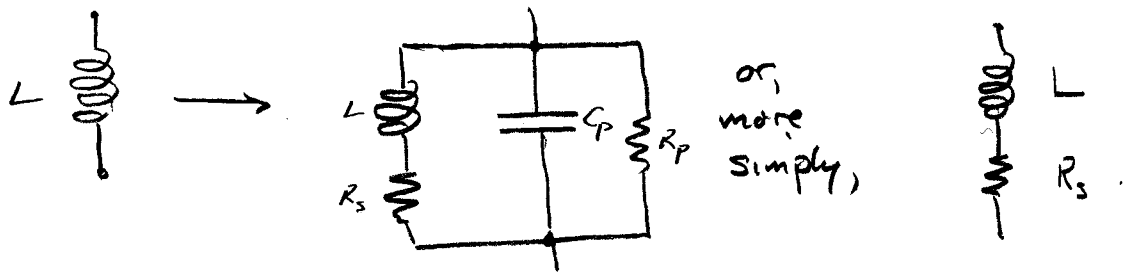
- Where does the energy go?

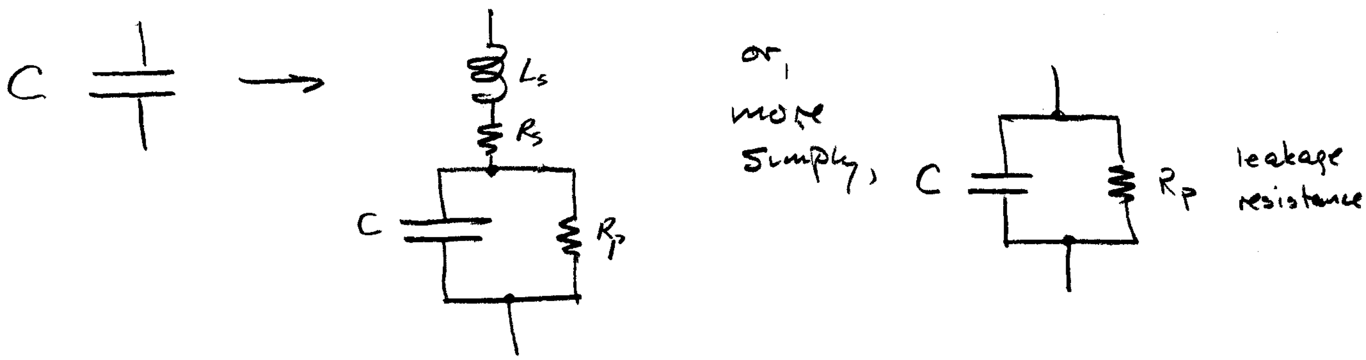
4.4.5 Imperfect Components | DC&L, sect 17.9

- Components don't behave exactly like our idealized models



but we can model <u>much</u> of the non-ideality by complexes of ideal components:





The extra elements are "parasitics". Which ones are important depends on the physical construction and frequencies of interest.

4.4.24

- To quantify the non ideality, note that any two-terminal impedance is

$$Z(j\omega) = \mathrm{Re}[Z(j\omega)] + j\,\mathrm{Im}[Z(j\omega)] = R(\omega) + jX(\omega)$$

For caps and inductors, we want just imaginary; real part is annoying. So define the quality factor

$$Q_{comp}(\omega) = \underbrace{\frac{X(\omega)}{R_s} = \frac{\omega L}{R_s}}_{\text{inductor}} \qquad Q_{cap}(\omega) = \frac{B(\omega)}{G_p} = \underbrace{\frac{\omega C}{G_p} = \omega R_p C}_{\text{cap}}$$

note frequency dependent

- <u>not</u> the same as Q of a tuned circuit.

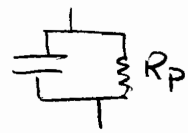
- here, the Q is the ratio $\dfrac{\text{desired}(\omega)}{\text{undesired}(\omega)}$

- For the simpler inductor model

$$Z = R_s + j\omega L$$

so $$Q_L(\omega) = \frac{\omega L}{R_s}$$

- For the simpler cap model C



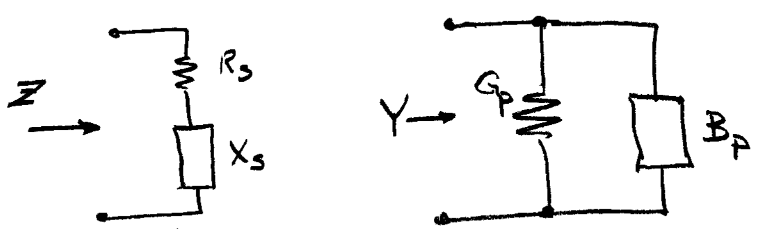
$$Q_c(\omega) = \omega R_p C \qquad \text{(see text)}$$

- Both better at higher frequencies (on physical grounds, too).

- what frequency to use? In a tuned circuit, use $\omega = \omega_0$, since that is the centre of the interesting frequency range.

- These models represent the imperfect components over a range of frequencies, but they certainly complicate circuit analysis:
  - the extra Ls and Cs add more poles and zeros;
  - the extra Rs shift the location of poles and zeros

- But simple estimates of the R effects are possible at or near a fixed frequency. Since ω doesn't change (much), a series or parallel model will work. Choose whichever makes life easier.
  - To transform series to parallel or the reverse

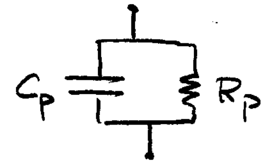


B: "susceptance"

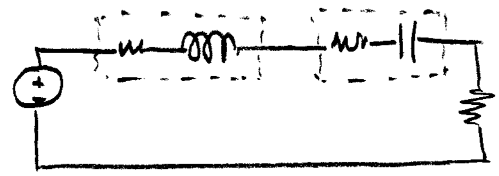
$Z = R_s + jX_s$ $\qquad\qquad$ $Y = G_p + jB_p$ $\qquad$ (for a given ω)

set $$Y = \frac{1}{Z} = \frac{1}{R_s + jX_s} = \frac{R_s}{R_s^2 + X_s^2} - j\frac{X_s}{R_s^2 + X_s^2} = G_p + jB_p$$

or $$Z = \frac{1}{Y} = \frac{1}{G_p + jB_p} = \frac{G_p}{G_p^2 + B_p^2} - j\frac{B_p}{G_p^2 + B_p^2} = R_s + jX_s$$

- Example: Transform the cap model  to a series model at a specific freq ω.

Why? Because it makes analysis of series RLC easier



We have $Y = G_p + j\omega C_p \qquad (G_p = 1/R_p)$

So equate $R_s + \frac{1}{j\omega C_s} = R_s - j\frac{1}{\omega C_s}$ to $\left(G_p + j\omega C_p\right)^{-1}$:

$$\frac{1}{Y} = \frac{1}{G_p + j\omega C_p} = \frac{G_p}{G_p^2 + \omega^2 C_p^2} - j\,\frac{\omega C_p}{G_p^2 + \omega^2 C_p^2} = \underbrace{\frac{R_p}{1+\omega^2 C_p^2 R_p^2}}_{R_s} - j\,\underbrace{\frac{\omega C_p R_p^2}{1+\omega^2 C_p^2 R_p^2}}_{1/\omega C_s}$$

Still messy, but recall

$$Q_c(\omega) = \omega R_p C_p \Rightarrow R_p = \frac{Q_c}{\omega C_p}$$

This makes

$$R_s = \frac{R_p}{1 + Q_c^2(\omega)} \underset{hi\,Q_c}{\approx} \frac{R_p}{Q_c^2(\omega)} = \frac{1}{Q_c\,\omega C_p}$$

$$C_s = \frac{1+\omega^2 C_p^2 R_p^2}{\omega^2 C_p R_p^2} = \frac{1+Q_c^2}{Q_c^2}\,C_p = C_p\left(1 + \frac{1}{Q_c^2}\right) \underset{hi\,Q_c}{\approx} C_p$$

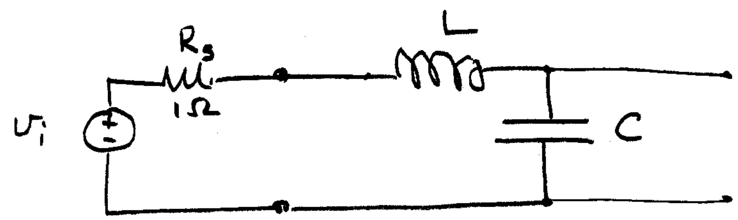
We now have a model in terms of the nominal C and its Q factor (= dissipation factor^{-1})

• *Example*: We want a series RLC circuit with $L = 20\ mH$, $C = 0.05\mu F$, to give $\omega_o = 31.62$ krad/s. At that frequency,

$$Q_L = 40 \qquad Q_c = 100 \quad (\text{equiv}, d_c = 0.01 \text{ diss factor}).$$

from manufacturer specs or earlier measurements. The source is buffered by an op amp with 1Ω output resistance. What is the minimum bandwidth we can obtain from this tuned circuit?

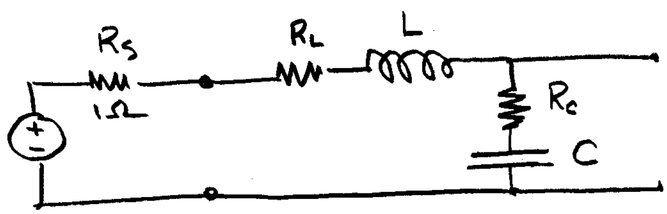
Idealized view:



$$H(s) = \frac{1/(LC)}{s^2 + \frac{R_s}{L}s + \frac{1}{LC}} \qquad \beta = \frac{R_s}{L} = 50 \text{ rad/s}$$

$$Q = \frac{31.62 \times 10^3}{50} = 632 \ !$$

Reality:



Find R_L, R_c

$$Q_L = \frac{\omega_o L}{R_L} \quad \text{so} \quad R_L = \frac{\omega_o L}{Q_L} = \underbrace{\sqrt{\frac{L}{C}}}_{632.4} \cdot \underbrace{\frac{1}{Q_L}}_{\frac{1}{40}} = 15.8\ \Omega$$

$$R_c = \underbrace{\frac{1}{Q_c \omega_o C_p}}_{\text{p. 4.4.26}} = \underbrace{\sqrt{\frac{L}{C}}}_{632.4} \underbrace{\frac{1}{Q_c}}_{\frac{1}{100}} = 6.32\ \Omega$$

So modified transfer function is

$$H(s) = \frac{s R_c/L + \frac{1}{LC}}{s^2 + \frac{R_t}{L} s + \frac{1}{LC}} \quad , \quad R_t = R_s + R_L + R_c$$

$$= 23.1\ \Omega$$

and the minimum bandwidth is

$$\beta_{min} = \frac{R_t}{L} = \frac{23.1}{20\times10^{-3}} = 1.16 \text{ krad/s}$$

$$Q_{max} = \frac{\omega_0}{\beta} = \frac{31.6}{1.16} = 27.2$$ a little more realistic

- De Carlo and Lin do a similar example with parallel models.