5.2 Butterworth Design

- Butterworth lowpass filters are all-pole filters of the form

$$H_B(s) = \frac{(-1)^n p_1 p_2 \cdots p_n}{(s-p_1)(s-p_2)\cdots(s-p_n)} \qquad H_B(0) = 1$$

- For a cutoff frequency (3 dB down frequency) $\omega_c = 1$ rad/s, the poles are equispaced about the unit circle.

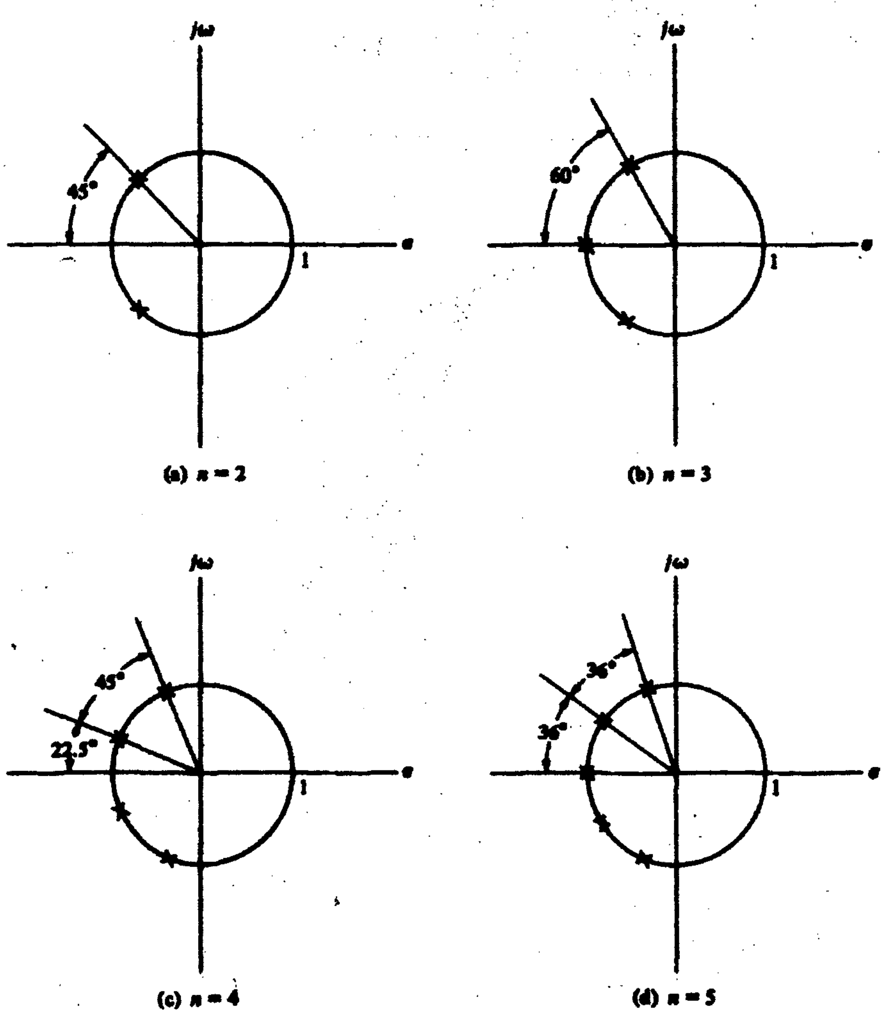


FIGURE D-3. Pole locations for normalized Butterworth filters.

– The poles are the $(2n)^{th}$ roots of $(-1)^{n-1}$.

– More generally, they are on a circle of radius ω_c

- Rewrite the filter as

$$H_B(s) = \frac{1}{B_n(s)} \longleftarrow \text{"Butterworth polynomial"}$$

They are easily obtained from the pole locations:

n	$B_n(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.61803s+1)(s^2+1.61803s+1)$
6	$(s^2+0.5176s+1)(s^2+\sqrt{2}s+1)(s^2+1.9319s+1)$

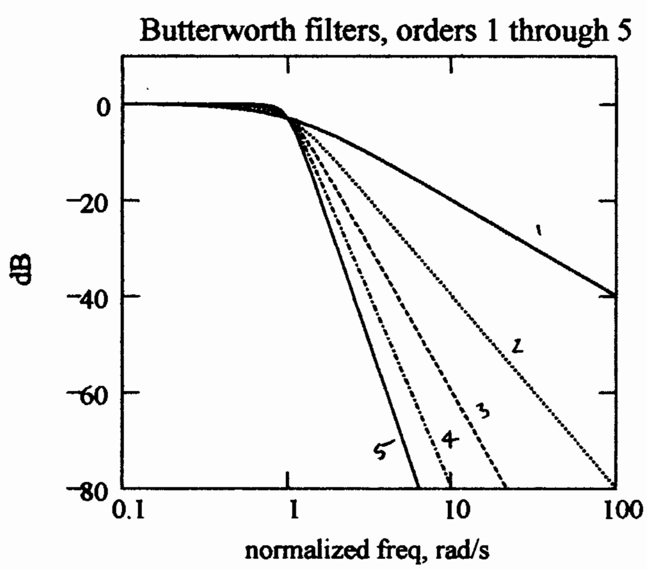
When expanded, Butterworth polys have palindromic coefficients, See p. 5.2.10

- The cutoff frequency plays a key role, since

$$|H_B(j\omega)|^2 = \frac{1}{1+(\omega/\omega_c)^{2n}} \qquad \text{very simple}$$

DC & L shows the Butterworth response to be "maximally flat – greatest number of derivatives $\frac{d^k}{d\omega^k}|H_B(j\omega)|^2$ are zero at $\omega = 0$.

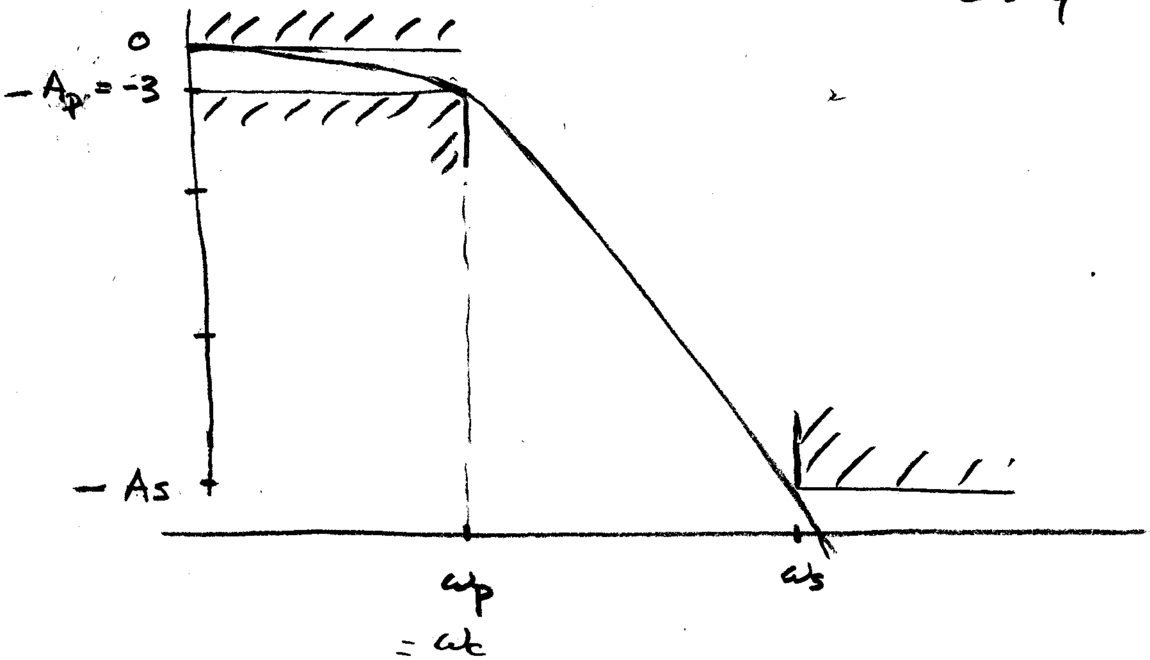
- Pretty good filters:



– How many dB/decade?

– Why?

- If the passband edge is selected to be the cutoff frequency, i.e. set $\omega_p = \omega_c$, then an easy design!



— To pick the order, select n so that

$$1 + \left(\frac{\omega_s}{\omega_p}\right)^{2n} \geq 10^{0.1 A_s}$$

or
$$n \geq \frac{\log(10^{A_s/10} - 1)}{2 \log(\omega_s/\omega_p)}$$

generally oversatisfies the criterion, since round up to integer.

— Armed with n, frequency scale the polynomial

$$H_B(s) = \frac{1}{B_n(s/\omega_p)}$$

- <u>example</u> Find a Butterworth lowpass with 3-dB bandwidth $\omega_p = \omega_c = 10$ krad/s and at least 40 dB attenuation above $\omega_s = 40$ krad/s.

  - Find the minimum order. $A_s = 40$. From p 5.2.3

$$n \geq \frac{\log(10^4 - 1)}{2\log(40/10)} = \frac{4}{2 \cdot 0.6} = 3.3$$

Choose $n = 4$

  - And the transfer function is

$$H(s) = \frac{1}{B_4(s/10^4)} =$$

$$= \frac{1}{\left(\frac{s}{10^4}\right)^4 + 2.613\left(\frac{s}{10^4}\right)^3 + 3.412\left(\frac{s}{10^4}\right)^2 + 2.613\frac{s}{10^4} + 1}$$

or, using the factored form in DC+L,

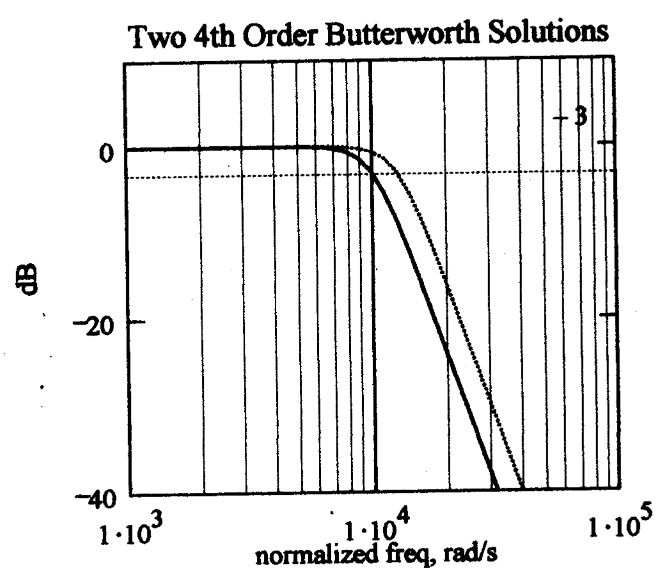
$$H(s) = \frac{1}{\left[\left(\frac{s}{10^4}\right)^2 + 0.765\frac{s}{10^4} + 1\right]\left[\left(\frac{s}{10^4}\right)^2 + 1.85\frac{s}{10^4} + 1\right]}$$

- Note that rounding up to $n=4$ gives some flexibility. We kept the 3 dB drop at ω_p and "over met" the 40dB requirement at ω_s. We could instead have increased ω_c so that we just met the 40 dB criterion, but reduced the attenuation (hence distortion) at the edge of the passband. Doing so, we find the new ω_c by

$$1 + \left(\frac{\omega_s}{\omega_c}\right)^8 = 10^{40/10} \qquad (2n=8)$$

or

$$\omega_c = \frac{\omega_s}{(10^4-1)^{1/8}} = 0.316\,\omega_s \qquad (\text{instead of } \tfrac{1}{4}\omega_s)$$



Choose according to which criterion is more important. Or run down the middle, to allow for component tolerances.

- If the passband edge is specified by a tighter spec than 3 dB, then $\omega_c \neq \omega_p$ and we have a more difficult design, with two params to choose: n and ω_c.

  - We have (ω_p, A_p) and (ω_s, A_s) to satisfy. Write them both:

$$1 + \left(\frac{\omega_p}{\omega_c}\right)^{2n} \leq 10^{A_p/10} \qquad \text{note } \leq$$

or

$$\left(\frac{\omega_p}{\omega_c}\right)^{2n} \leq 10^{A_p/10} - 1 \qquad (1)$$

similarly

$$\left(\frac{\omega_s}{\omega_c}\right)^{2n} \geq 10^{A_s/10} - 1 \qquad \text{note } \geq \qquad (2)$$

  - Divide (2) by (1) to eliminate ω_c

$$\left(\frac{\omega_s}{\omega_p}\right)^{2n} \geq \frac{10^{0.1 A_s} - 1}{10^{0.1 A_p} - 1} \qquad \text{note } \geq \text{, tricky}$$

or

$$n \geq \frac{\log\left(\frac{10^{0.1 A_s} - 1}{10^{0.1 A_p} - 1}\right)}{2 \log(\omega_s / \omega_p)}$$

the criterion for filter order (round up)

— Having found n, we turn to ω_c. As on p. 5.2.5, we have some freedom: we can satisfy A_p exactly and oversatisfy A_s in the stop band; we can meet A_s at ω_s exactly, and over meet A_p at ω_p; or something in between.

- Exact at passband edge:

$$1 + \left(\frac{\omega_p}{\omega_c}\right)^{2n} = 10^{A_p/10} \qquad \text{note} =$$

or

$$\omega_c = \frac{\omega_p}{\left(10^{0.1 A_p} - 1\right)^{1/2n}}$$

- Exact at stopband edge

$$1 + \left(\frac{\omega_s}{\omega_c}\right)^{2n} = 10^{A_s/10}$$

or

$$\omega_c = \frac{\omega_s}{\left(10^{A_s/10} - 1\right)^{1/2n}}$$

— With n and ω_c determined, the filter is

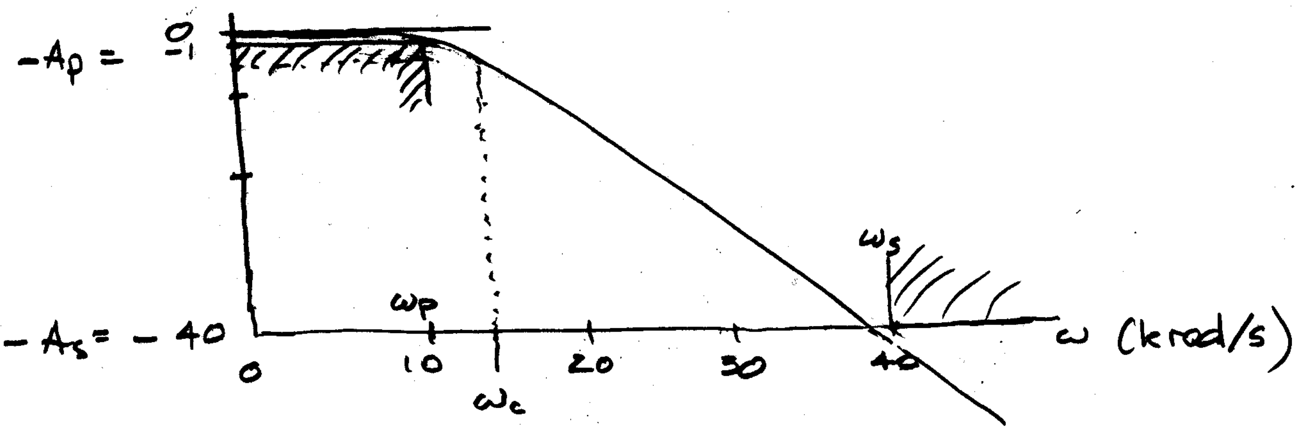
$$H_B(j\omega) = \frac{1}{B_n(s/\omega_c)}$$

An example on p. 5.3.10.

- **Example** Design a lowpass filter with max 1 dB variation in the passband, defined by $\omega_p = 10$ krad/s and at least 40 dB attenuation above $\omega_s = 40$ krad/s. Use a Butterworth design.

  - Summary: $\omega_p = 10$ krad/s $\quad A_p = 1$ dB
    $\omega_s = 40$ krad/s $\quad A_s = 40$ dB

  - Sketch the requirements:



  - What filter order lets us meet both specs?

$$n = \frac{\log\left(\frac{10^{0.1A_s}-1}{10^{0.1A_p}-1}\right)}{2\log(\omega_s/\omega_p)} = \frac{\log\left(\frac{10^{4}-1}{10^{0.1}-1}\right)}{2\log\left(\frac{40}{10}\right)} = 3.809$$

round up to

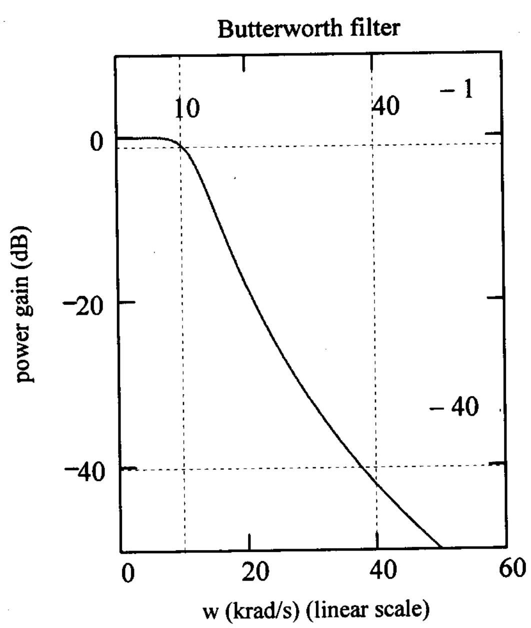
$n = 4$

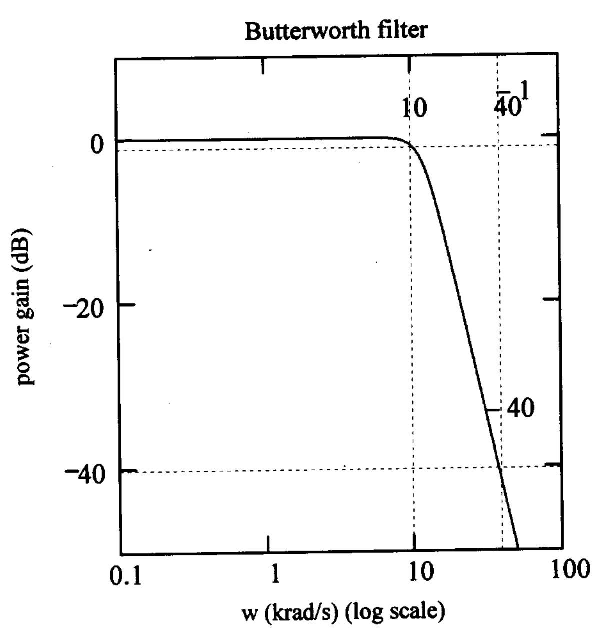
p 5.2.6

- We have some freedom in choice of ω_c:
  - graze the passband corner, or the stopband corner, or something in between;
  - my choice – satisfy passband with equality.
- The cutoff frequency (p. 5.2.7) is then

$$\omega_c = \frac{\omega_p}{\left(10^{0.1A_p}-1\right)^{1/2n}} = \frac{10\text{ krad/s}}{\sqrt[8]{10^{0.1}-1}} = 11.84\text{ krad/s}$$

- So power "gain" is $|H_B(j\omega)|^2 = \dfrac{1}{1+\left(\dfrac{\omega}{11.84\times10^3}\right)^8}$





- What about the group delay? For Butterworth,

$$D_n(\omega) = \frac{1}{1+\omega^{2n}} \sum_{m=0}^{n-1} \frac{\omega^{2m}}{\sin(\pi(2m+1)/2n)}$$

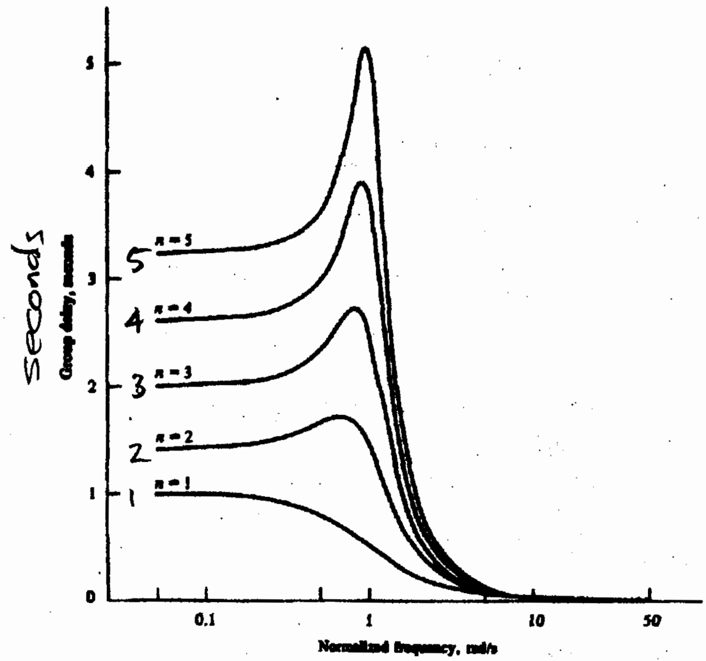


FIGURE D-5. Group delay of Butterworth filters.

*See Weinberg (1962), p. 497.

– The curves are for $\omega_c = 1$ rad/sec

– By time-freq scaling, the delay for other cutoff freqs is

$$\frac{D_n(\omega/\omega_c)}{\omega_c}$$

– Distortion if signal spans varying delays

- And that's all, for our pass at Butterworth design

And, for completeness, the Butterworth polys expanded.

TABLE D-1
Butterworth Polynomials $1 \le n \le 6$[a]

$B_1(s) = s + 1$
$B_2(s) = s^2 + 1.4142136s + 1$
$B_3(s) = s^3 + 2.0000000s^2 + 2.0000000s + 1$
$B_4(s) = s^4 + 2.6131259s^3 + 3.4142136s^2 + 2.6131259s + 1$
$B_5(s) = s^5 + 3.2360680s^4 + 5.2360680s^3 + 5.2360680s^2 + 3.2360680s + 1$
$B_6(s) = s^6 + 3.8637033s^5 + 7.4641016s^4 + 9.1416202s^3 + 7.4641016s^2 + 3.8637033s + 1$

[a] $H_n(s) = 1/B_n(s)$.