5.3 Chebyshev I Design

- Chebyshev I (or just "Chebyshev") filters have ripple in the passband and rapidly increasing attenuation in the transition band (sharper cutoff than Butterworth).

- An all-pole filter like Butterworth, so they both drop asymtotically as 20 n dB/decade, although Chebyshev always has more atten. than Butterworth.

- From Tedeschi:

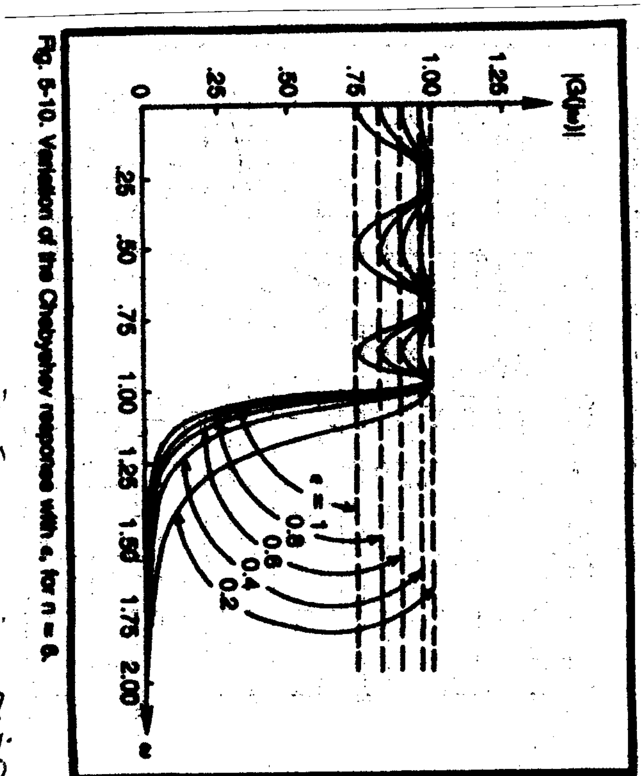


Fig. 5-10. Variation of the Chebyshev response with ε, for n = 6.

faster drop with increasing ripple

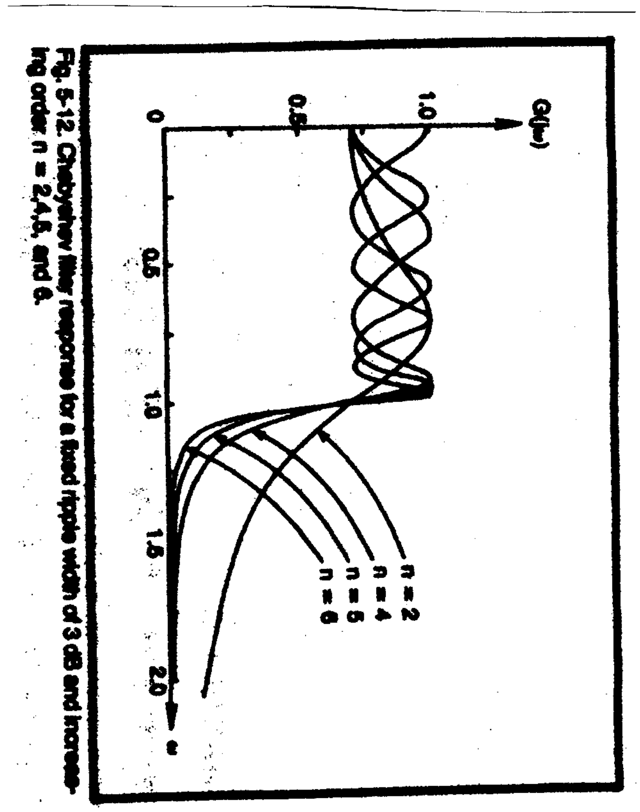


Fig. 5-12. Chebyshev filter response for a fixed ripple width of 3 dB and increasing order: n = 2,4,6, and 8.

number of passband extrema = order n

• You won't need to use the fact, but Cheby I poles lie on an ellipse.

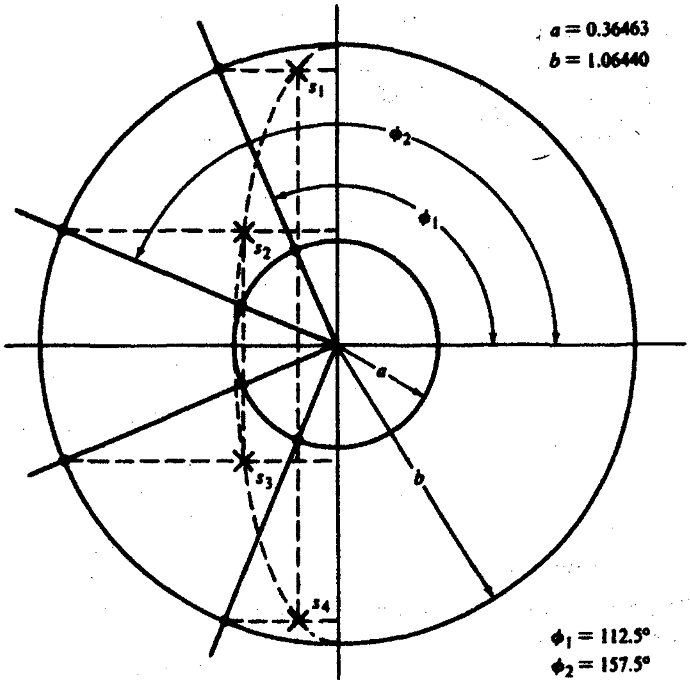
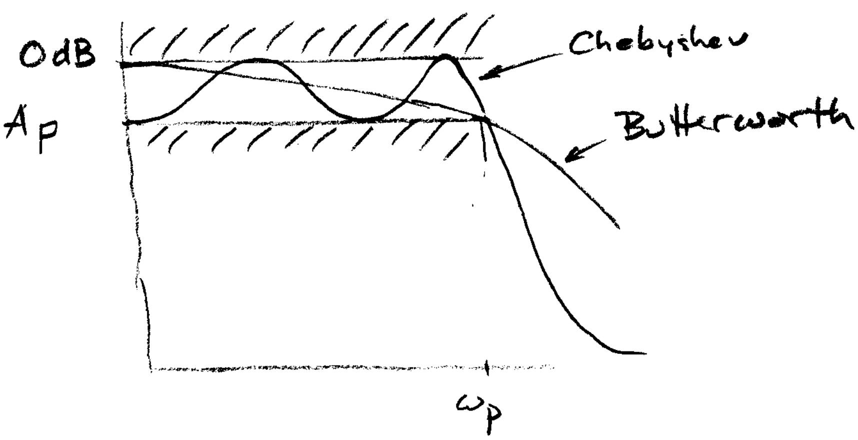


FIGURE D-10. Pole locations for a fourth-order normalized Chebyshev I filter with 1-dB passband ripple.

It's more common to use tables of polynomials than to construct the roots,

or "Poles + polynomials of Chebyshev I filters" in Supplementary Notes and Demos section of website

• In the design, we have two parameters to choose: passband ripple and filter order. The ripple is analogous to Butterworth A_p at ω_p.



- We determine the tradeoff between ripple and filter order by the expression for power gain of the Cheby I filter:

$$|H_c(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)} \qquad (\text{for } \omega_p = 1 \text{ rad/s})$$

where $C_n(\omega)$ is the Chebyshev polynomial of degree n:

$$C_n(\omega) = \cos\left(n \cos^{-1}(\omega)\right), \quad |\omega| \leq 1$$

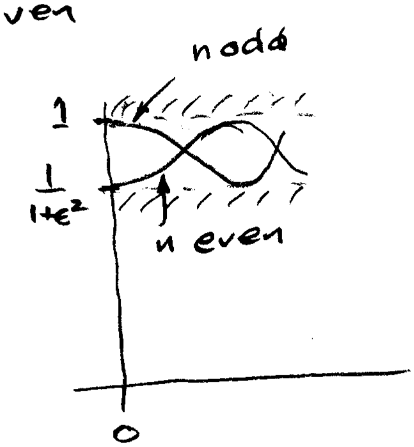
- It's not obvious that this is a polynomial, so see Appendix 5A.

- Within the passband, $C_n(\omega)$ oscillates between -1 and 1, since this polynomial is a cosine with a warped axis:

→ DC gain? At $\omega = 0$, $\cos^{-1}(\omega) = \pi/2$ and

$$C_n(0) = \cos\left(n \frac{\pi}{2}\right) = \begin{cases} 0, & n \text{ odd} \\ \pm 1, & n \text{ even} \end{cases}$$

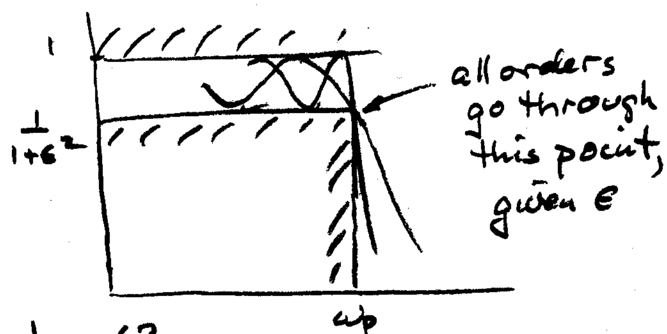
so $$|H_c(0)|^2 = \begin{cases} 1, & n \text{ odd} \\ \frac{1}{1+\epsilon^2}, & n \text{ even} \end{cases}$$



→ Gain at passband edge? For $\omega = 1$ rad/s,

$\cos^{-1}(1) = 0$, $C_n(1) = 1$ and

$$|H_c(j)|^2 = \frac{1}{1+\epsilon^2}$$



→ Number of extrema in passband?

As ω runs 0 to 1, $\theta = \cos^{-1}(\omega)$ runs $\pi/2$ down to 0.

So $\cos^2(n\theta)$ has n extrema (don't count band edge, $\theta = 0$)

→ Relate ϵ to A_p

$$10 \log\left(\frac{1}{1+\epsilon^2}\right) = -A_p$$

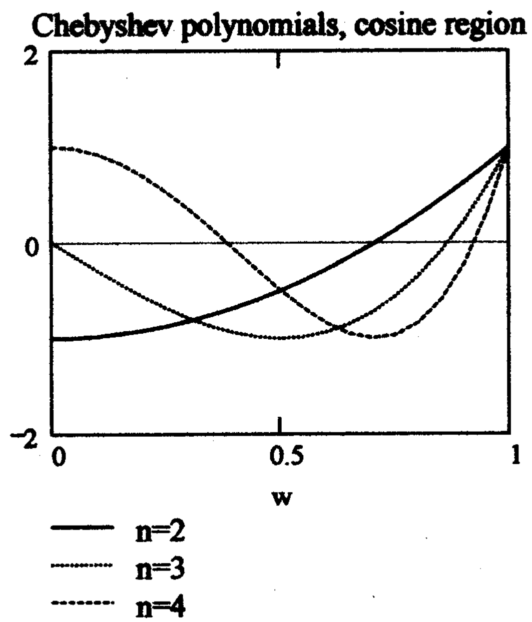
$(0.1 \ln(10) = 0.23)$

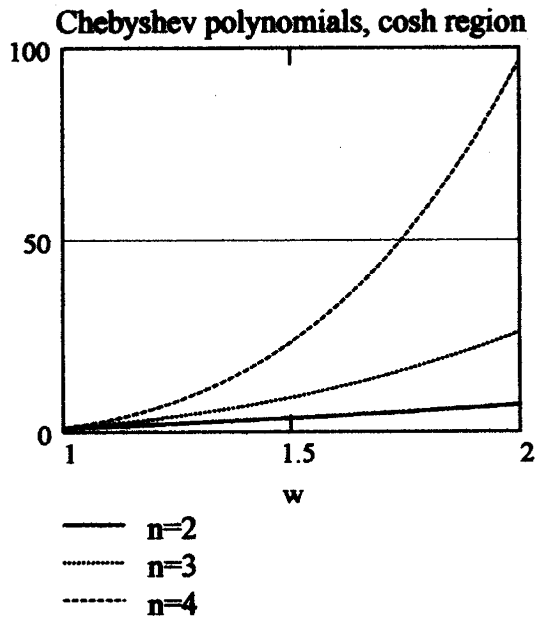
$$\epsilon^2 = 10^{0.1 A_p} - 1 = e^{0.23 A_p} - 1 \approx 0.23 A_p$$

- Outside the passband, $\omega > 1$, the arc cos is problematic: $\cos^{-1}(\omega)$ is imaginary. It and the outer cos are more easily expressed as cosh:

$$C_n(\omega) = \cosh(n \cosh^{-1}(\omega)), \quad |\omega| > 1$$

So, over a larger domain, the polynomials look like this



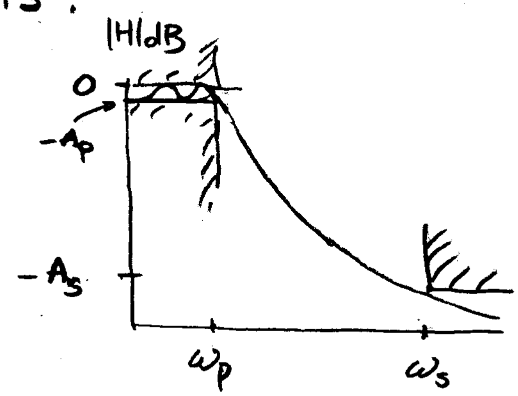


- To scale to a non-unit passband edge ω_p,

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega/\omega_p)}$$

- Now we can select filter order according to stop band requirements:

$$dB\left(|H(j\omega_s)|^2\right) \leq -A_s$$



or $1 + \epsilon^2 C_n^2(\omega_s/\omega_p) \geq 10^{0.1 A_s}$

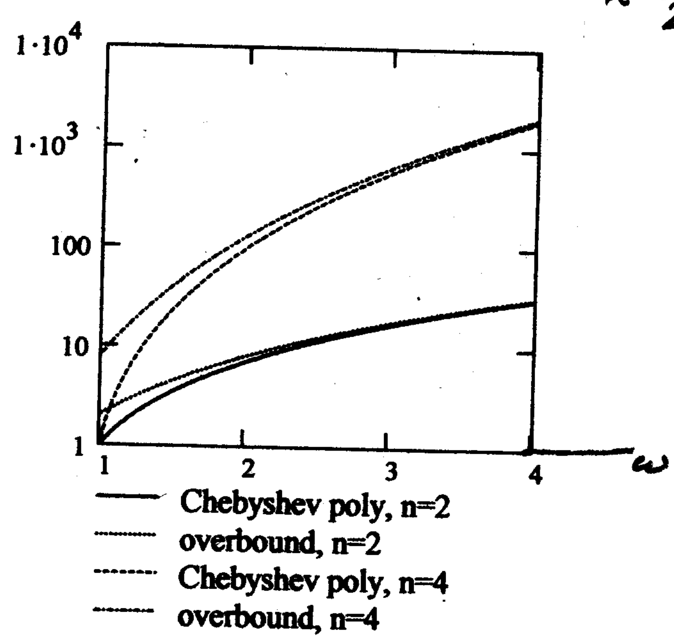
or $C_n(\omega_s/\omega_p) \geq \frac{1}{\epsilon}\sqrt{10^{0.1A_s} - 1}$

Since we have ω_s/ω_p, ϵ and A_s, increase n until the inequality is satisfied.

- There is no quick way to get n, although we can easily underbound it:
  - Since $\theta = \cosh^{-1}(\omega)$
    we have $\omega = \cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta}) \geq \frac{1}{2}e^{\theta}$
    Thus $\theta \leq \ln(2\omega)$ (a rough bound)
  - Similarly, $C_n(\omega) = \cosh(n\theta)$

$$\leq \cosh(n \ln(2\omega))$$

$$\approx \frac{1}{2} e^{n \ln(2\omega)} = \frac{1}{2}(2\omega)^n$$



- If we use the overbound, we know that n can be no less than the value it provides. Go from

$$C_n(\omega_s/\omega_p) \geq \frac{1}{\epsilon}\sqrt{10^{0.1A_s}-1}$$

to

$$\frac{1}{2}(2\omega_s/\omega_p)^n \geq \frac{1}{\epsilon}\sqrt{10^{0.1A_s}-1}$$

and

$$n \geq \frac{\ln\left(\frac{2}{\epsilon}\sqrt{10^{0.1A_s}-1}\right)}{\ln(2\omega_s/\omega_p)}$$

This not a bad estimate, since we rarely go for $\omega_s/\omega_p < 2$. From the graph, it's close.

- And now, the filter polynomials themselves:

Table 38.2 First six Chebyshev polynomials

n	A_{max}	denominator $D(s)$ $\quad H(s) = \frac{num}{D(s)}$	num
1	0.5 (dB)	$s+2.8628$	2.8628
2		$s^2+1.4256s+1.5162$	1.4314
3		$(s^2+0.6246s+1.1425)(s+0.6265)$	0.7157
4		$(s^2+0.3507s+1.0635)(s^2+0.8467s+0.3564)$	0.3579
5		$(s^2+0.2239s+1.0358)(s^2+0.5863s+0.4768)(s+0.3623)$	0.1789
6		$(s^2+0.1553s+1.0230)(s^2+0.4243s+0.5901)(s^2+0.5796s+0.1570)$	0.0948
1	1.0 (dB)	$s+1.9652$	1.9652
2		$s^2+1.0977s+1.1025$	0.9826
3		$(s^2+0.4942s+0.9942)(s+0.4942)$	0.4913
4		$(s^2+0.2791s+0.9865)(s^2+0.6737s+0.2794)$	0.2457
5		$(s^2+0.1789s+0.9883)(s^2+0.4684s+0.4293)(s+0.2895)$	0.1228
6		$(s^2+0.1244s+0.9907)(s^2+0.3398s+0.5577)(s^2+0.4641s+0.1247)$	0.0689

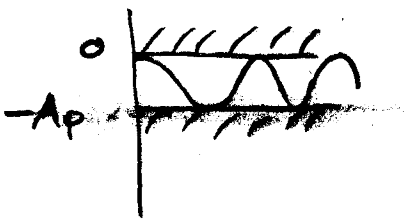
Select a filter according to A_p and n. Frequency scale with ω_p.

- **Example.** Design a lowpass filter with max 1 dB variation in the passband, defined by $\omega_p = 10$ krad/s and at least 40 dB attenuation above $\omega_s = 40$ krad/s. Use Cheby I.

  - So $A_p = 1$ dB, which makes

$$\epsilon \leq \sqrt{10^{A_p/10} - 1} = 0.51$$

Make it an equality, so the ripple really does graze the allowable A_p



$\epsilon = 0.51$

  - Next, estimate required filter order. It's at least

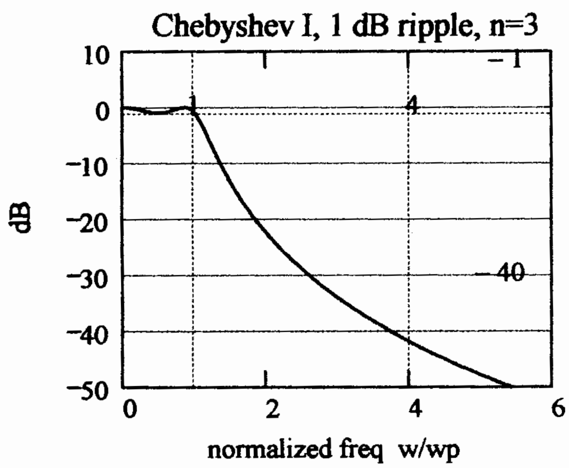
$$n \geq \frac{\ln\left(\frac{2}{\epsilon}\sqrt{10^{A_s/10} - 1}\right)}{\ln(2\omega_s/\omega_p)}, \quad A_s = 40\,\text{dB}$$

$$= 2.873$$

So try to get by with $n = 3$. If it fails, $n = 4$ will work.

- Plot the power gain for $n=3$ using the expression from p 5.3.5

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega/\omega_p)}$$



Met the spec with room to spare!

- The transfer function from the table, frequency scaled, is

$$H(s) = \frac{0.4913}{\left\{\left(\frac{s}{10^4}\right)^2 + 0.4942\left(\frac{s}{10^4}\right) + 0.9942\right\}\left\{\frac{s}{10^4} + 0.4942\right\}}$$

- Continue the example by comparing with a Butterworth that meets the same specs.

  - From p. 5.2.8,

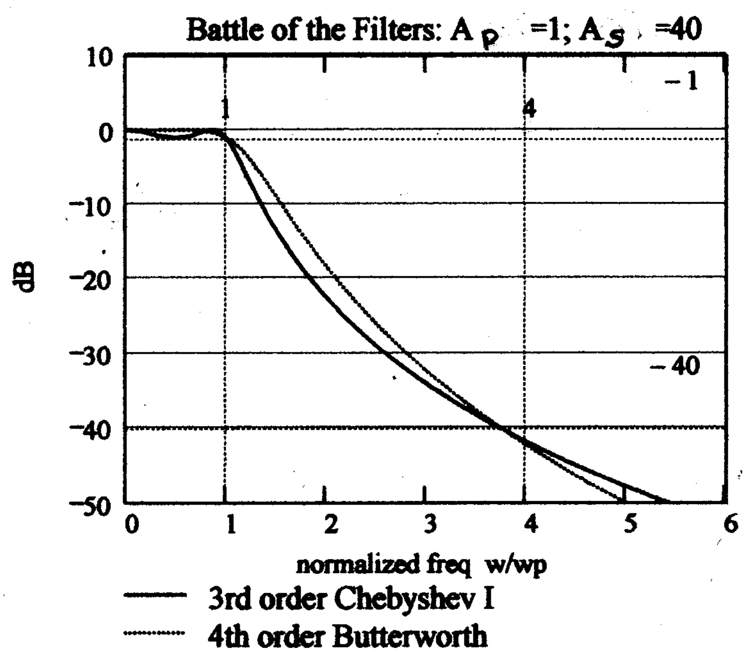
$$n \geq \frac{\log\left(\frac{10^{0.1 A_s} - 1}{10^{0.1 A_p} - 1}\right)}{2 \log(\omega_s/\omega_p)} = 3.809 \longrightarrow 4$$

So we need a 4^{th} order Butterworth, where a 3^{rd} order Cheby worked.

Its cutoff frequency, p 5.2.9, is

$$\omega_c = \frac{\omega_p}{\left(10^{A_p/10} - 1\right)^{1/2n}} = 11.84 \text{ krad/s}$$

and power gain is $|H_B(j\omega)|^2 = \dfrac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$



- They both meet the pass band spec.
- Cheby $n=3$ drops faster than Butter. $n=4$ in transition band
- They both beat spec in stop band
- Further out in stop band, Butterworth $n=4$ drops faster than Cheby $n=3$ because it has 4 poles, not 3.

Appendix About Chebyshev polynomials.

- Warp the ω axis with $\theta = \cos^{-1}(\omega)$. The functions $C_n(\omega) = \cos(n\theta)$ are the Chebyshev polynomials. Why are they polynomials?
  - For $n=1$, $C_1(\omega) = \cos(\cos^{-1}(\omega)) = \omega$
  - For $n=2$, $C_2(\omega) = \cos(2\theta)$

$$= 2\cos^2(\theta) - 1$$

$$= 2C_1^2(\omega) - 1 = 2\omega^2 - 1$$

Both are polynomials.

- For higher n, trig identity

$$\cos(n\theta) = 2\cos(\theta)\cos((n-1)\theta) - \cos((n-2)\theta)$$

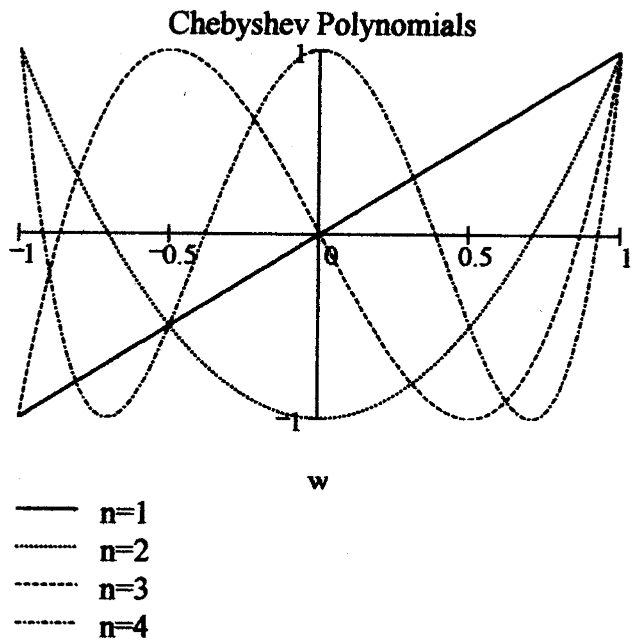
gives the recursion

$$C_n(\omega) = 2\omega C_{n-1}(\omega) - C_{n-2}(\omega)$$

which produces a polynomial of degree n.

$$C_1(\omega) = \omega$$
$$C_2(\omega) = 2\omega^2 - 1$$
$$C_3(\omega) = 4\omega^3 - 3\omega$$
$$C_4(\omega) = 8\omega^4 - 8\omega^2 + 1$$
$$\vdots$$

- Because Chebyshev polys can be expressed as a cosine, they have an equiripple property in $-1 \leq \omega \leq 1$



This property is very useful when we want to approximate a function over a given interval and minimize the maximum error — $C_n(\omega)$'s don't peak up in places; they are always bounded by $[-1, 1]$