5.6 State Variable Filters

- State variable filters are another approach to active filters. Advantages:
  - easy to design lowpass, bandpass, highpass bandstop, etc
  - accurate pole placement
  - manageable range of component values
  - can obtain zeros, as well as poles, for a general biquadratic

$$\frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

Disadvantages:

- they need at least twice as many op amps as Sallen-Key designs do

- State variable filters are directly linked to linear system theory. Given a rational polynomial H(s), the corresponding state variable structure is straightforward.

- System Diagrams

- We want a block diagram structure to represent any system that can be described by a linear DE, such as

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t)$$
$$= b_m x^{(m)}(t) + \cdots + b_0 x(t)$$

- Strategy:

→ Devise a structure for the same LHS coeffs, but the simplest RHS

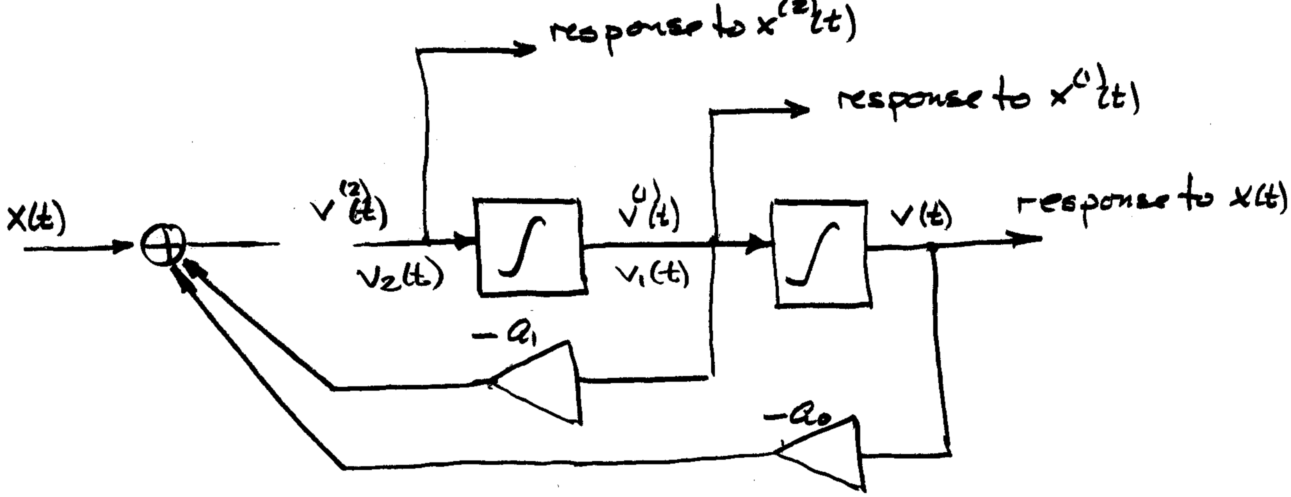
$$v^{(n)}(t) + a_{n-1} v^{(n-1)}(t) + \cdots + a_0 v(t) = x(t)$$

→ From it, obtain responses to $x^{(1)}(t)$, $x^{(2)}(t)$, etc.

→ and then the response to the full RHS

$$b_m x^{(m)}(t) + \cdots + b_0 x(t)$$

— For example, 2nd order



→ Input to highest integrator is

$$v^{(2)}(t) = -a_1 v^{(1)}(t) - a_0 v(t) + x(t)$$

So the structure satisfies the modified DE

$$v^{(2)}(t) + a_1 v^{(1)}(t) + a_0 v(t) = x(t).$$

→ By the derivative property, $v_1(t)$ is the solution of $v_1^{(2)}(t) + a_1 v_1^{(1)}(t) + a_0 v_1(t) = x^{(1)}(t)$

since $v_1(t) = v^{(1)}(t)$.

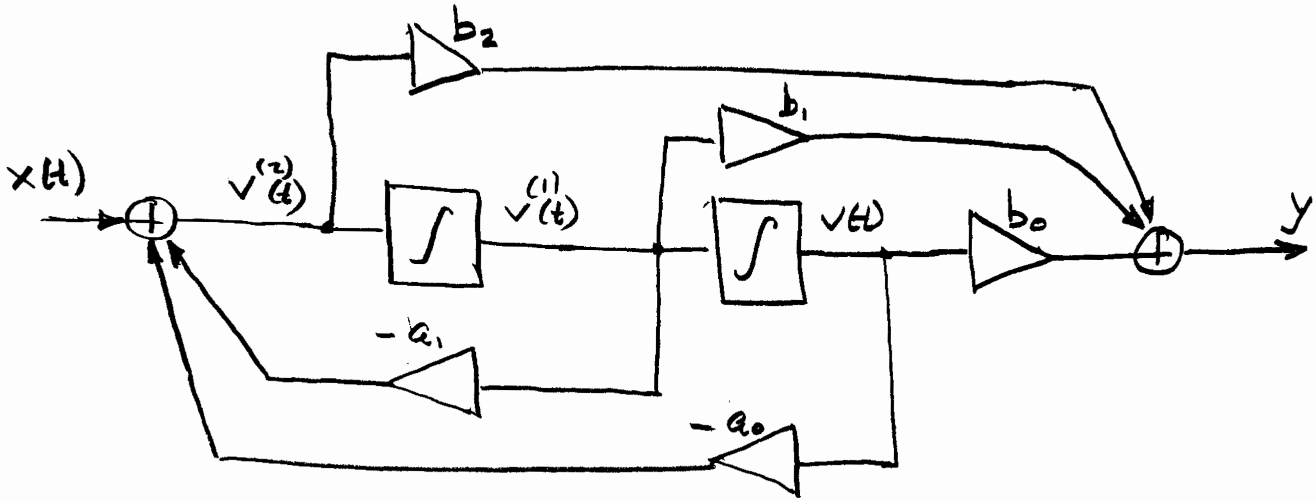
→ Similarly, $v_2(t)$ is the solution to

$$v_2^{(2)}(t) + a_1 v_2^{(1)}(t) + a_0 v_2(t) = x^{(2)}(t).$$

→ And $b_2 v^{(2)}(t) + b_1 v^{(1)}(t) + b_0 v(t) = y(t)$ is solution to

$$y^{(2)}(t) + a_1 y^{(1)}(t) + a_0 y(t) = b_2 x^{(2)}(t) + b_1 x^{(1)}(t) + b_0 x(t)$$

→ So the solution structure is



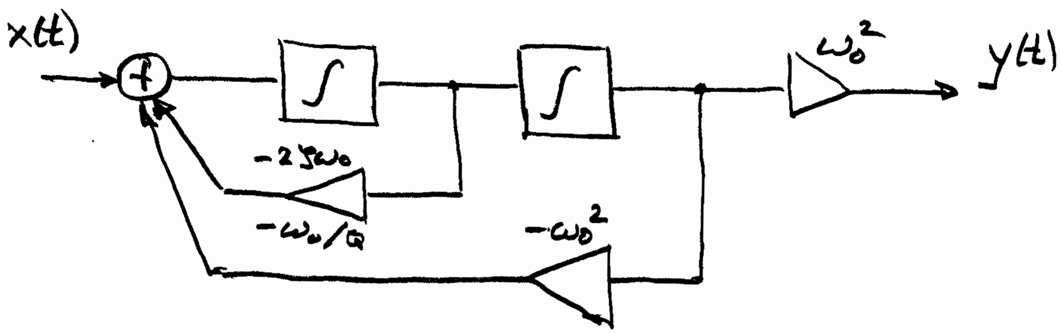
$$y^{(2)} + a_1 y^{(1)}(t) + a_0 y(t) = b_2 x^{(2)}(t) + b_1 x^{(1)}(t) + b_0 x(t)$$

It ~~implements~~ represents

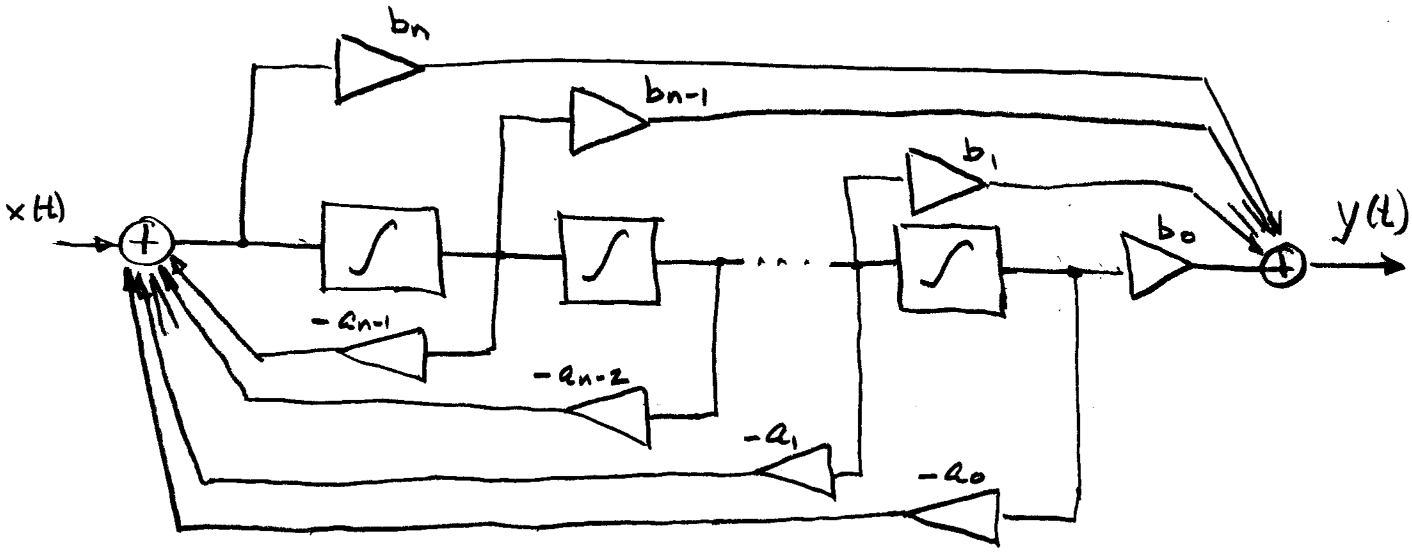
$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

a general "biquadratic" transfer function.

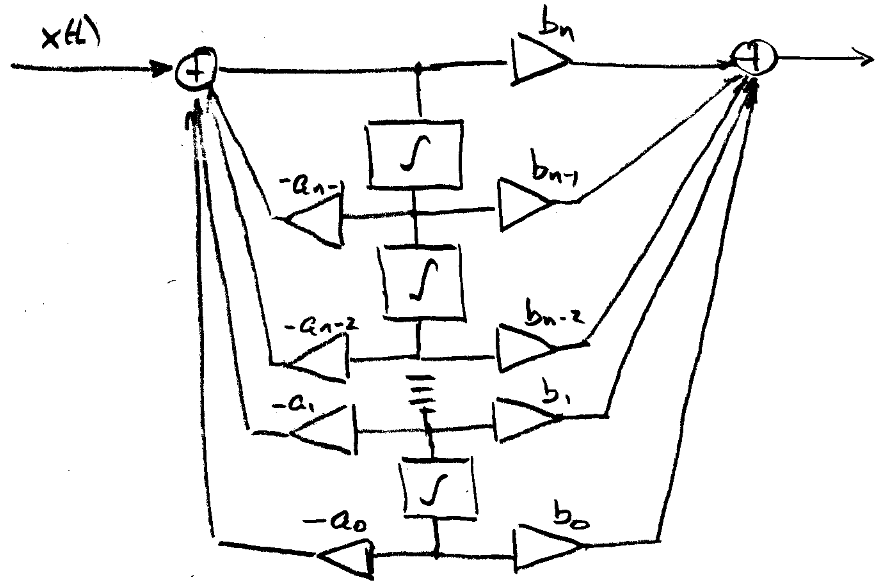
– <u>example</u> lowpass 2nd order $H(s) = \dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$



— Higher order is similar



Also drawn

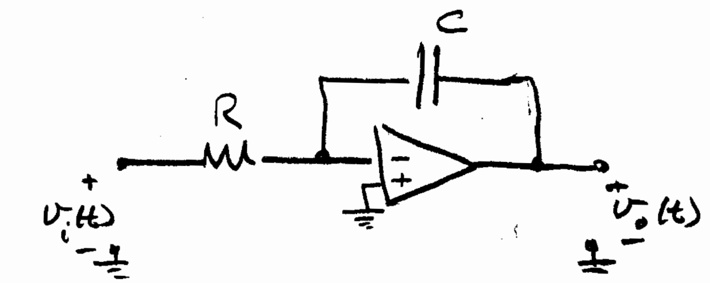


- Every linear system — mechanical, thermal, electric circuit, etc — has a system diagram like this.
- Why "state variable"? See Appendix to this section.

- **Wiring Up the System Diagram**

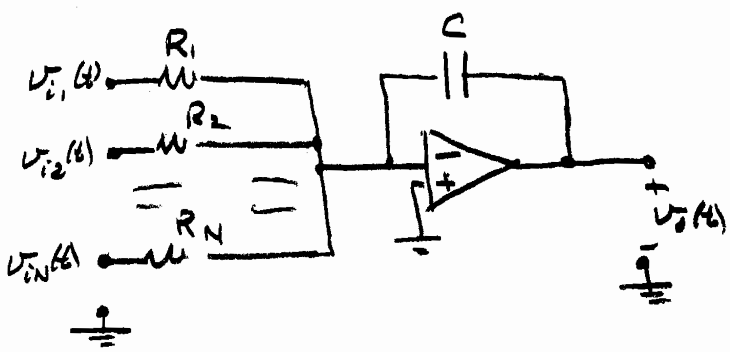
  - If the system diagram can represent any transfer function, why not just implement it to realize our filter? We have implementations for each of its blocks:

integrator (inverting)



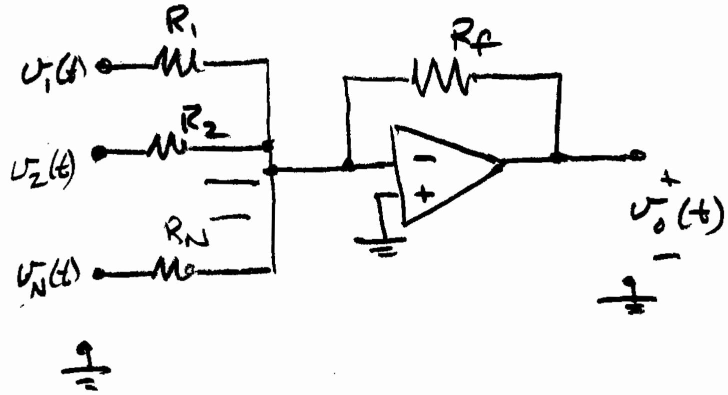
$$v_o(t) = -\frac{1}{RC}\int_{0^-}^{t} v_i(\alpha)\,d\alpha$$

summing integrator (inverting)



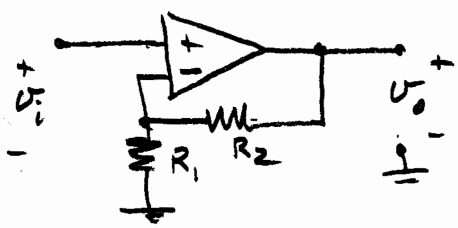
$$v_o(t) = -\int_{0^-}^{t}\left(\frac{1}{R_1C}v_{i1}(\alpha) + \frac{1}{R_2C}v_{i2}(\alpha) + \cdots + \frac{1}{R_NC}v_{iN}(\alpha)\right)d\alpha$$

summing amp (inverting)



$$v_o(t) = -\frac{R_f}{R_1}\, v_1(t) \cdots\cdots - \frac{R_f}{R_N}\, v_N(t)$$

gain block (non-inverting, gain ≥ 1)



$$v_o = \left(1 + \frac{R_2}{R_1}\right) v_i$$

- advantages of this approach:
  - → straight forward
  - → control of many parameters by simple adjustments (e.g. change Q with single resistor)
  - → lowpass, bandpass, highpass, bandstop
- drawbacks
  - → more op amps: one per integrator, so at least two for a quad section; sign inversion sometimes requires another op amp to correct.

- *example* 3rd order Butterworth lowpass

→ From p. 5.2.10

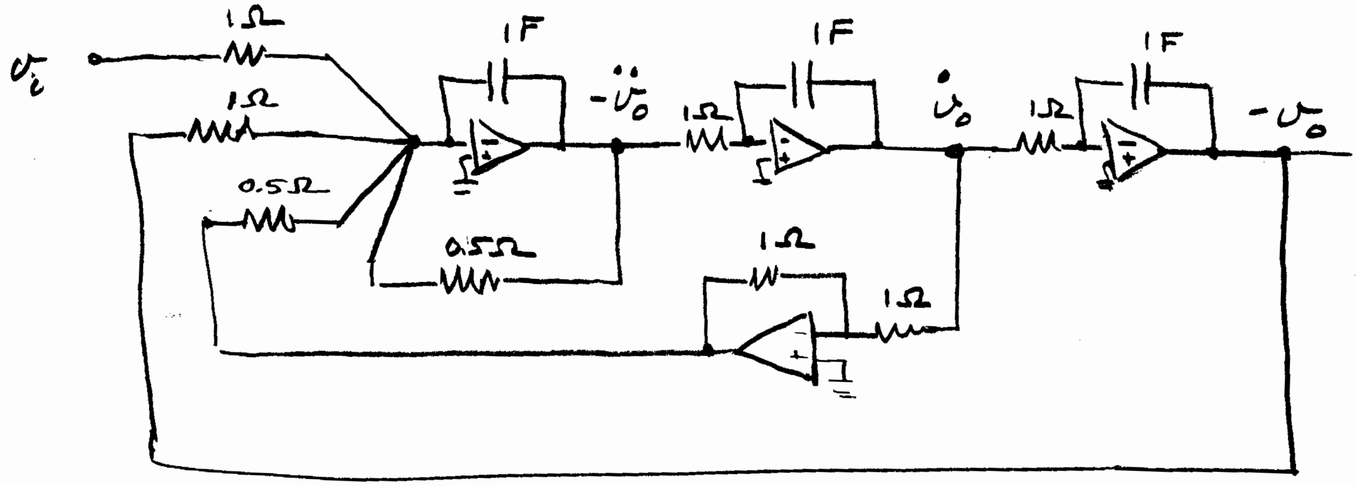
$$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$ for $\omega_c = 1$ rad/s

so

$$\dddot{v}_o + 2\ddot{v}_o + 2\dot{v}_o + v_o = v_i$$

or

$$\dddot{v}_o = -2\ddot{v}_o - 2\dot{v}_o - v_o + v_i$$



→ frequency scale, impedance scale as usual

→ four op amps, compared with Sallen-Key's two op amps

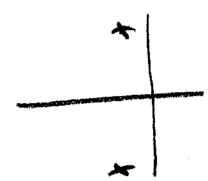
- State Variable Quadratic Sections

— Even though we can implement a filter of any order with this approach, it is still more common to use quadratic sections. Why? Less sensitivity to coefficient error. In fact, even DSP implementations use quad sections.

— Here is a quad section that makes lowpass, bandpass, highpass outputs available

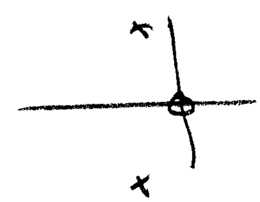
$$H_{lp}(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

$$= \frac{1}{s^2 + \frac{1}{Q} s + 1}$$ (normalized)

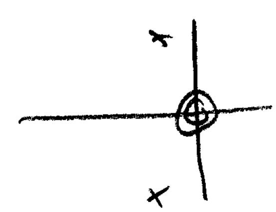


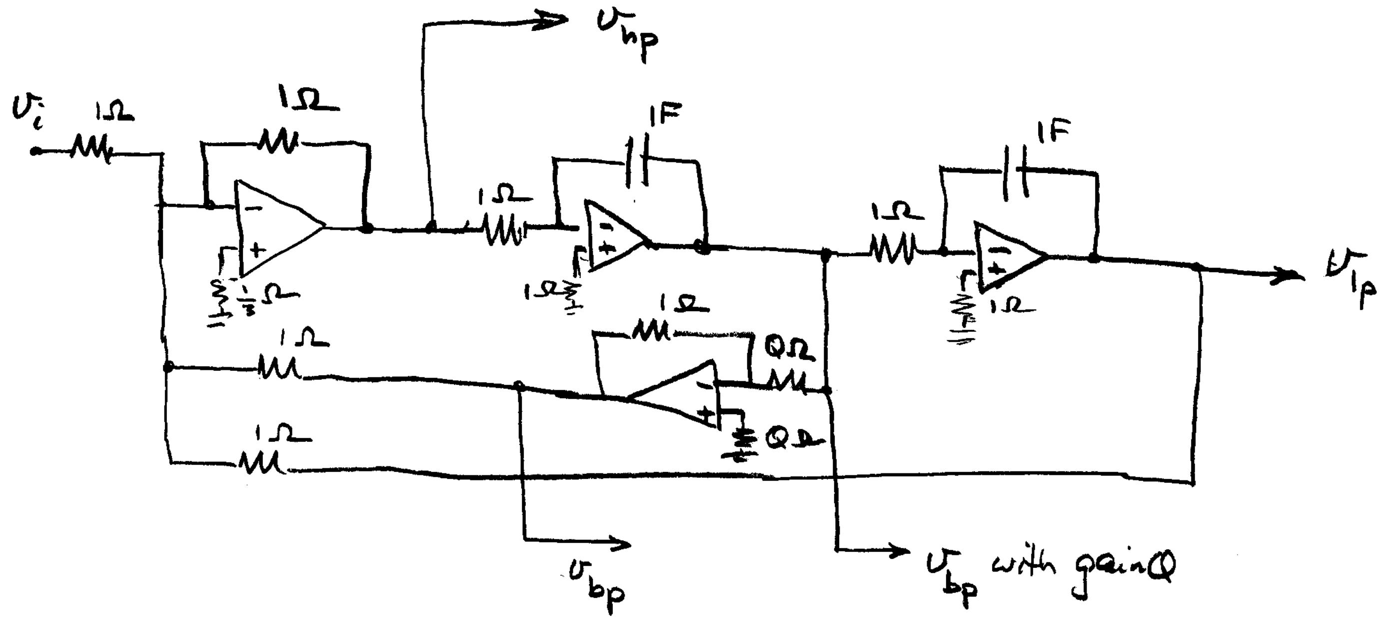
other numerators

$$H_{bp}(s) = \frac{s/Q}{s^2 + \frac{1}{Q} s + 1}$$



$$H_{hp}(s) = \frac{s^2}{s^2 + \frac{1}{Q} s + 1}$$





→ control Q with one resistor (two if offset an issue)

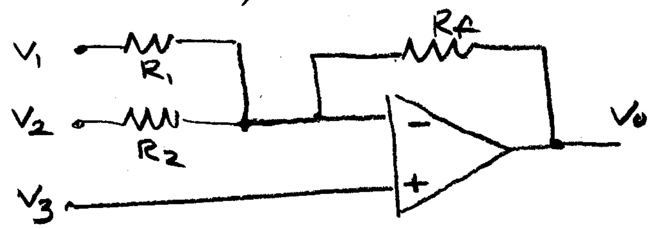
→ control overall gain with input resistor

→ bandpass output with gain 1 or gain Q at resonance

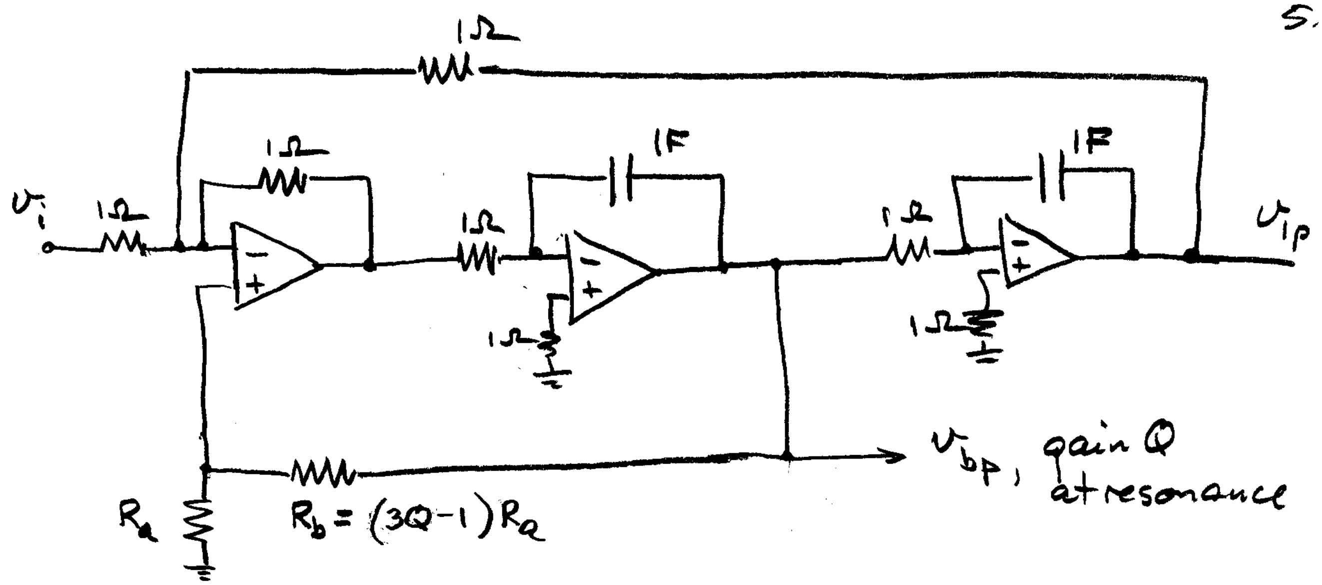
→ all three elementary numerators available.

— Sometimes we don't need all that flexibility — for example, if bandpass output with gain Q is what we want.

→ First, a subcircuit



$$V_o = -\frac{R_f}{R_1}V_1 - \frac{R_f}{R_2}V_2 + \left(1 + \frac{R_f}{R_1 \| R_2}\right)V_3$$



→ voltage divider ratio is $\frac{1}{3Q}$ to compensate gain of 3 at "+"

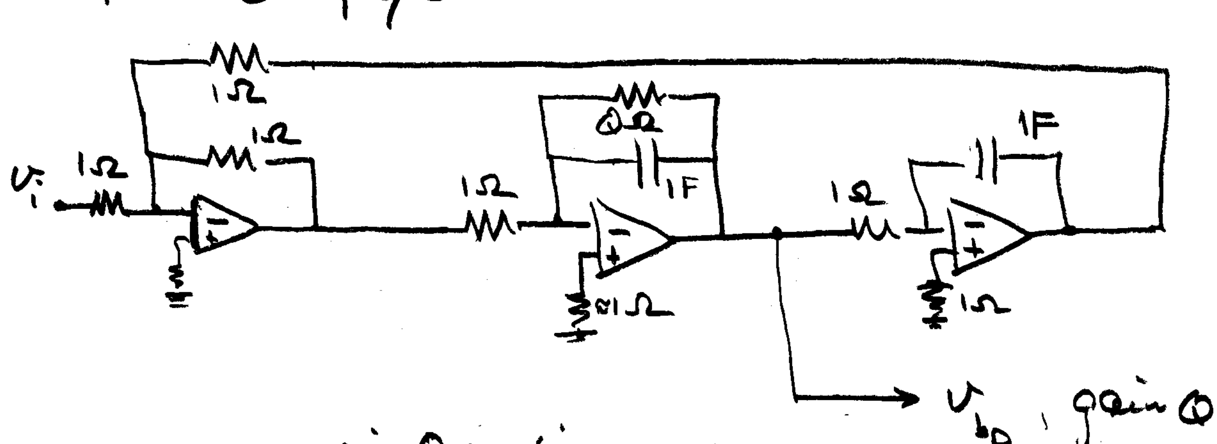
→ should have $R_a \| R_b = \frac{1}{3}\Omega$ for min offset, since resistance seen by "−" terminal is $\frac{1}{3}\Omega$.

Hence $R_a = \frac{1}{3}\ \frac{3Q}{3Q-1}\ \Omega \approx \frac{1}{3}\ \Omega$

$$R_b = (3Q-1)R_a = Q\ \Omega$$

→ note gain and Q are coupled: change input resistor for different gain and must change R_b/R_a, too.

– or, more simply still:



→ no gain, Q coupling

→ but no highpass output

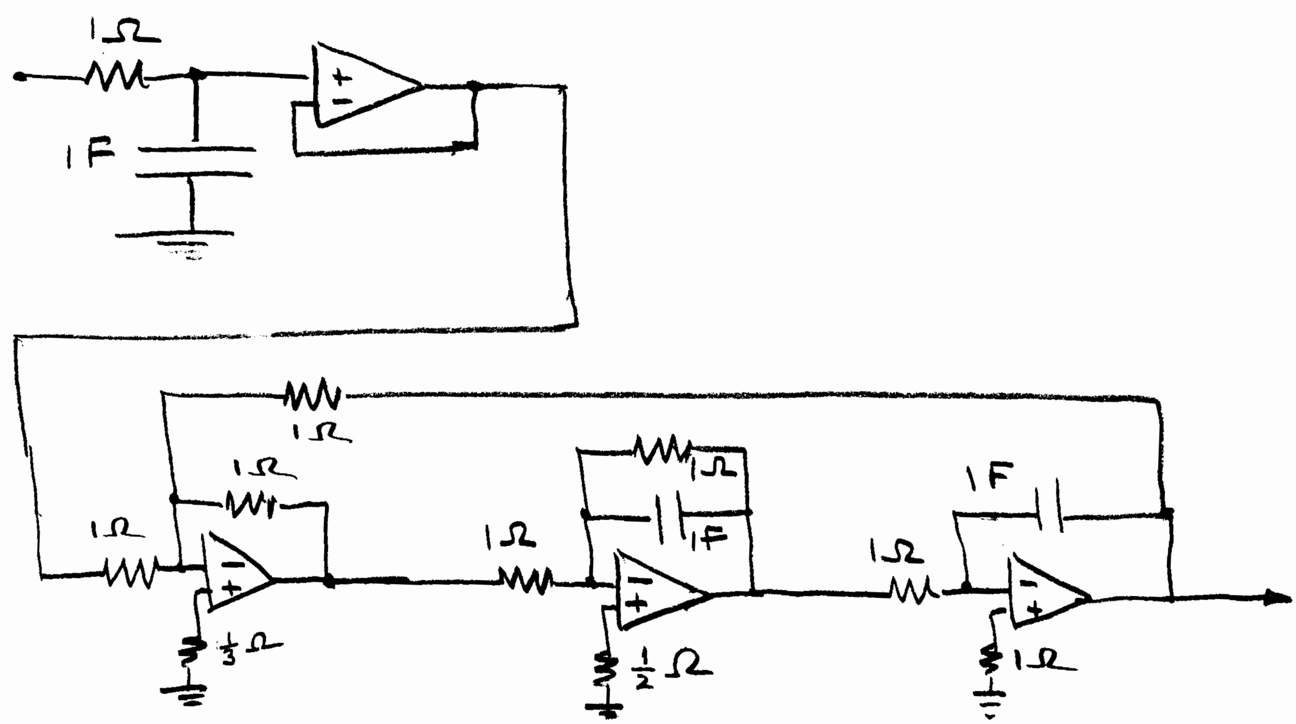
5.6.12

• example Design a 3rd order Butterworth lopass with ω_c = 10 krad/s using a state variable quad section (and a linear one).

$$H(s) = \frac{1}{s^3+2s^2+2s+1} = \frac{1}{(s+1)(s^2+s+1)}$$

quadratic section has $Q = 1$

Normalized filter



— Frequency scale factor $K_f = 10^4$, divide caps by K_f

— Impedance scale $K_i = 1000$, for 1 kΩ resistors, further reduction in C's

— So all caps 0.1 μF, all resistors 1 kΩ except the 500 Ω resistor for bias current.