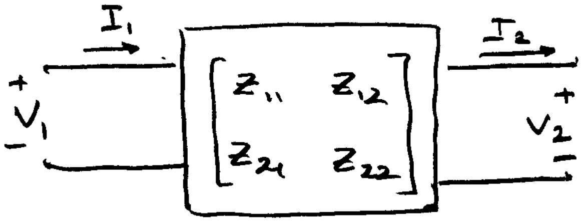
7.3 Impedance Parameter Analysis

DC+L 19.5, 19.6

- It is equally common to characterize two ports by impedance parameters ("z parameters")



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

or

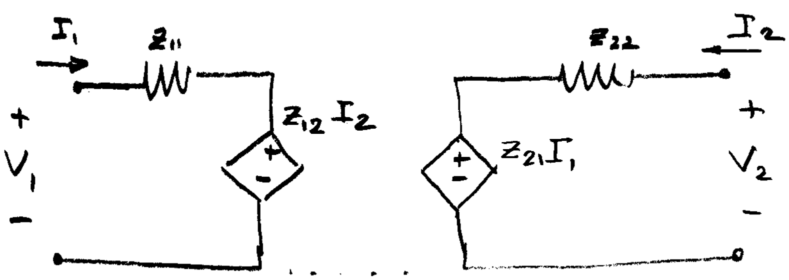
$$V_1 = z_{11} I_1 + z_{12} I_2$$
$$V_2 = z_{21} I_1 + z_{22} I_2$$

where the elements are "open circuit impedances"

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

As with y params, the z parameters can be obtained by circuit analysis or by measurements on a physical circuit.



- From the analysis of y-parameters in Section 7.2, you know what sort of questions to ask:
  - How to calculate the z parameters from a circuit diagram?
  - How to calculate input impedance, output impedance and voltage gain when the two port is terminated?

and you know the style of analysis.

We'll do some examples and develop counterpart results for z-parameters – but first, a question.

- Why not obtain z parameters from y-parameters?

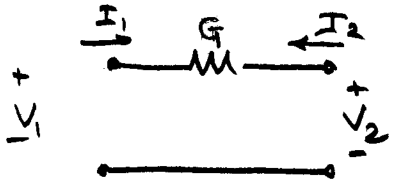
If
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

then don't we have
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

provided the inverse exists?

Yes, but the inverse might not exist.

- This circuit has y-parameters

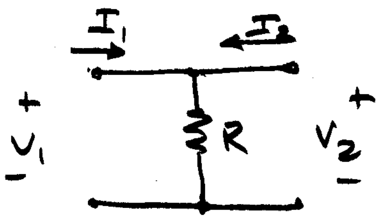


$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} G & -G \\ -G & G \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

but $\begin{bmatrix} G & -G \\ -G & G \end{bmatrix}^{-1}$ does not exist, so no easy route to z params — which don't exist.

e.g. $z_{11} = \left.\dfrac{V_1}{I_1}\right|_{I_2=0} = ?$

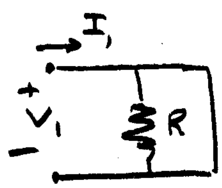
- And this circuit has z-parameters



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R & R \\ R & R \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

but $\begin{bmatrix} R & R \\ R & R \end{bmatrix}^{-1}$ does not exist, so no y-params

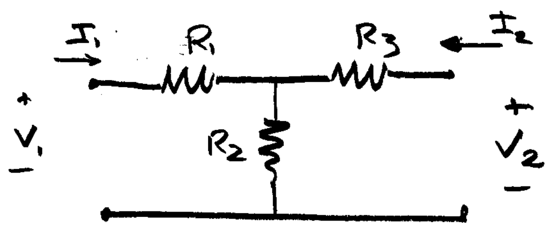
e.g. $y_{11} = \left.\dfrac{I_1}{V_1}\right|_{V_2=0} = ?$



- These are unusual circuits — most have both y and z params — but they demonstrate limits of these ideas.

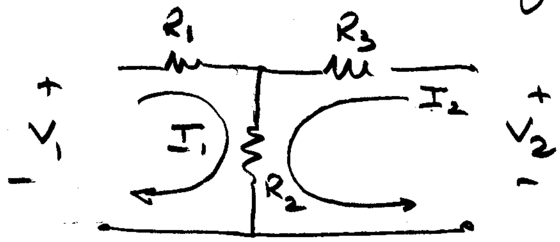
- Back to calculation of z parameters. They lend themselves better to loop analysis.

- <u>example</u> simple T circuit like the one on p. 7.2.2.



It's easy to get the open circuit impedances by inspection.

We'll do it by loop equations



The R's can be replaced by any other circuit element for something a little different.

$$V_1 = R_1 I_1 + R_2 (I_1 + I_2)$$

$$V_2 = R_3 I_2 + R_2 (I_2 + I_1)$$

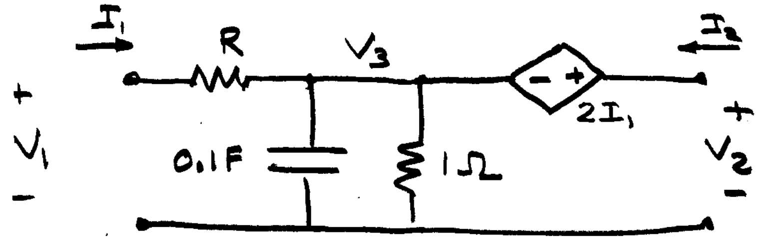
or

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 + R_2 & R_2 \\ R_2 & R_2 + R_3 \end{bmatrix}}_{\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

symmetric, since no dependent sources.

$\boxed{\text{Read DC+L ex. 19.10}}$

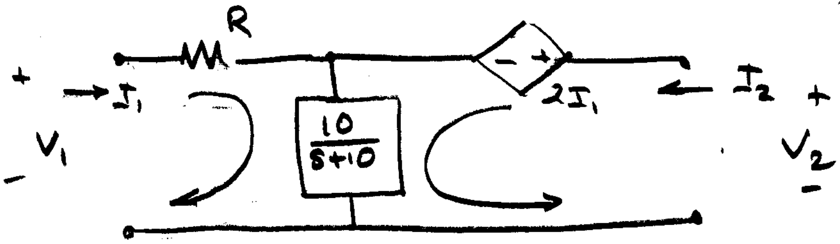
- example with a dependent source (also see DC+L ex 19.9)



Nodal analysis is easy, but it gives $\begin{bmatrix} I_1 \\ \vdots \\ J \end{bmatrix} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_2 \end{bmatrix}$

We could invert the matrix...

Instead, we'll do loop analysis, to get z params directly. First, combine the parallel C, R to remove one loop ($1\Omega \| 0.1F \longrightarrow \frac{10}{s+10}$)



$$V_1 = RI_1 + \frac{10}{s+10}(I_1 + I_2)$$

$$V_2 = 2I_1 + \frac{10}{s+10}(I_2 + I_1)$$

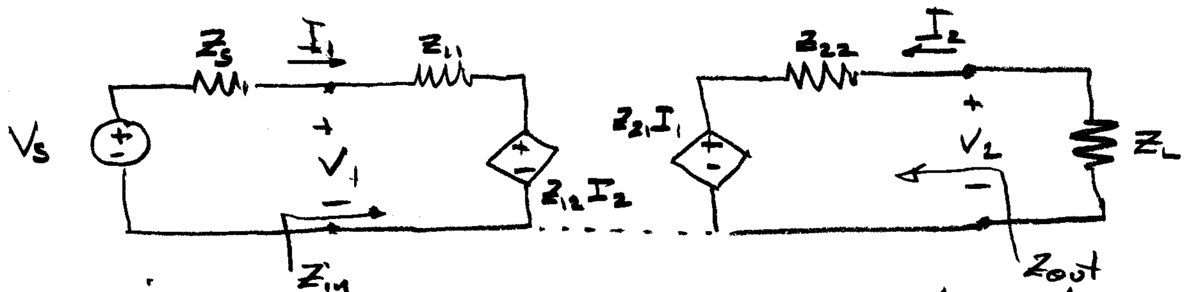
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} R + \frac{10}{s+10} & \frac{10}{s+10} \\ 2 + \frac{10}{s+10} & \frac{10}{s+10} \end{bmatrix}}_{\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

not symmetric – dependent source

- Now move on to input and output impedances.

$\boxed{\text{DC+L 19.6}}$

- The derivation is much like that of input, out admittances with y parameters. The surprising point is similarity of the results.



input impedance

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

and from KVL on RHS of circuit

$$I_2 = - \frac{Z_{21} I_1}{Z_{22} + Z_L}$$

substitution of I_2 into 1^{st} eq'n gives

$$V_1 = \left(Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22} + Z_L} \right) I_1$$

so

$$Z_{in} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22} + Z_L}$$

output impedance

$$V_2 = Z_{22} I_2 + Z_{21} I_1$$

and from KVL on LHS with independent source set to zero

$$I_1 = - \frac{Z_{12} I_2}{Z_{11} + Z_s}$$

substitution of I_1 into first eq'n gives

$$V_2 = \left(Z_{22} - \frac{Z_{12} Z_{21}}{Z_{11} + Z_s} \right) I_2$$

so

$$Z_{out} = Z_{22} - \frac{Z_{12} Z_{21}}{Z_{11} + Z_s}$$

Again, note pattern of subscripts — their similarity, and their similarity with the y-parameter counterparts.

- We can generalize to impedance <u>or</u> admittance (call it "immitance"). $\boxed{\text{DC+L 19.8}}$

Denote by $\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$ — they could be impedance or admittance, depending on context or choice

Then

$$\Gamma_{in} = \gamma_{11} - \frac{\gamma_{12}\gamma_{21}}{\gamma_{22} + \Gamma_L} \qquad \Gamma_{out} = \gamma_{22} - \frac{\gamma_{12}\gamma_{21}}{\gamma_{11} + \Gamma_s}$$

or even

$$\Gamma_{port} = \gamma_{near,near} - \frac{\gamma_{near\,far}\,\gamma_{far\,near}}{\gamma_{far\,far} + \Gamma_{far}}$$

- The pattern carries through to h-params, too.

- Voltage gain is obtained as before.

$$G_v = \frac{V_2}{V_1} \cdot \frac{V_1}{V_s} = G_{v_2} G_{v_1}$$

$$V_2 = \frac{Z_L}{Z_L + z_{22}} \cdot z_{21} I_1 \quad \text{(voltdiv.)}$$

$$= \underbrace{\frac{Z_L}{Z_L + z_{22}}}_{G_{v2}} \cdot \frac{z_{21} V_1}{Z_{in}}$$

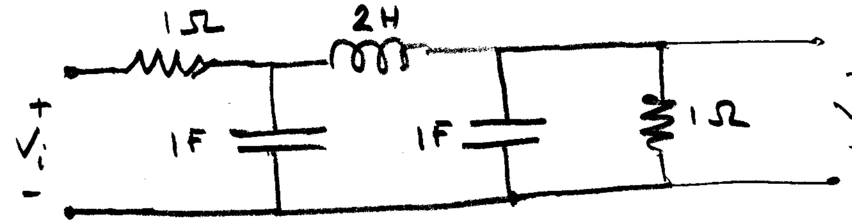
$$V_1 = \underbrace{\frac{Z_{in}}{Z_{in} + Z_s}}_{G_{v_1}} \cdot V_s \quad \text{(volt. div)}$$

$$G_v = \frac{Z_L}{Z_L + z_{22}} \cdot \frac{z_{21}}{Z_{in} + Z_s}$$

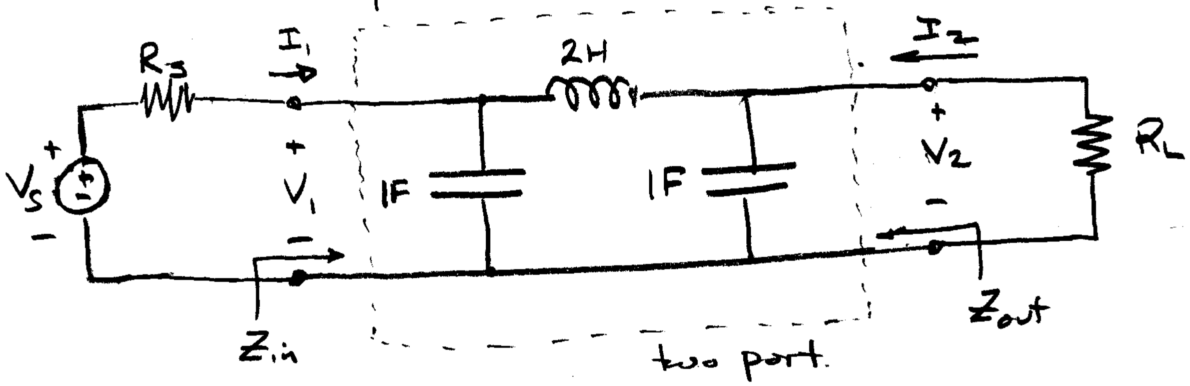
not similar to G_v with y params

$\boxed{\text{Read DC+L example 19.12}}$

• **big example** Treat the passive 3^{rd} order Butterworth lowpass from p. 5.4.2 as a two-port. The circuit is

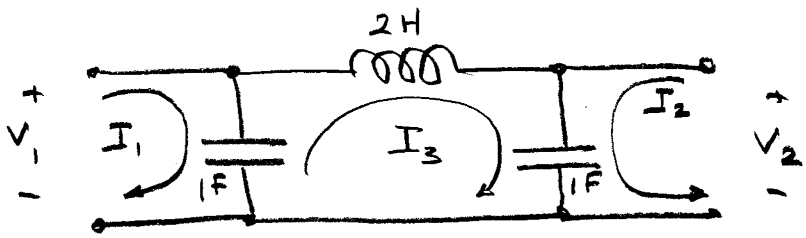


We know that source and load resistances will alter the behaviour of the circuit, so this example will modify the circuit to



We want
- the z parameters
- the input and output impedances
- the voltage gain (which should be Butterworth with 1Ω source and load resistances.)

– Start with the z parameters. Use loop equations.



$$V_1 = \frac{1}{s} I_1 \qquad\qquad - \frac{1}{s} I_3$$

$$V_2 = \qquad \frac{1}{s} I_2 + \frac{1}{s} I_3$$

$$0 = -\frac{1}{s} I_1 + \frac{1}{s} I_2 + \left(\frac{2}{s} + 2s\right) I_3$$

$$\begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & 0 & -\frac{1}{s} \\ 0 & \frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s} + 2s \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

We don't want I_3, so eliminate it.

Solve the "non interface" equation for the unwanted I_3:

$$I_3 = -\frac{s}{2(s^2+1)} \begin{bmatrix} -\frac{1}{s} & \frac{1}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

and substitute in the "wanted block"

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{s} \\ \frac{1}{s} \end{bmatrix} \frac{s}{2(s^2+1)} \begin{bmatrix} -\frac{1}{s} & \frac{1}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$= \underbrace{\frac{1}{2s(s^2+1)} \begin{bmatrix} 2s^2+1 & 1 \\ 1 & 2s^2+1 \end{bmatrix}}_{\text{the z parameters}} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

– Now for Z_{in}, Z_{out}:

$$Z_{in}(s) = Z_{11} - \frac{Z_{12}Z_{21}}{Z_{22}+Z_L} = \frac{2s^2+1}{2s(s^2+1)} - \frac{\left(\frac{1}{2s(s^2+1)}\right)^2}{\frac{2s^2+1}{2s(s^2+1)} + R_L}$$

$$= \frac{2R_L s^2 + 2s + R_L}{2R_L s^3 + 2s^2 + 2R_L s + 1}$$

and by symmetry

$$Z_{out}(s) = \frac{2R_s s^2 + 2s + R_s}{2R_s s^3 + 2s^2 + 2R_s s + 1}$$

– Finally, the voltage gain V_2/V_s:

$$G_v = \frac{R_L}{R_L + Z_{22}} \cdot \frac{Z_{21}}{Z_{in}+R_s}$$

$$= \frac{R_L}{R_L + \frac{2s^2+1}{2s(s^2+1)}} \cdot \frac{\frac{1}{2s(s^2+1)}}{\frac{2R_L s^2+2s+R_L}{2R_L s^3+2s^2+2R_L s+1} + R_s}$$

$$= \frac{R_L}{2R_sR_L s^3 + 2(R_s+R_L)s^2 + 2(1+R_sR_L)s + (R_s+R_L)}$$

and if $R_s = R_L = 1\Omega$,

$$G_v = \frac{1}{2}\,\frac{1}{s^3+2s^2+2s+1} \qquad 3^{rd} \text{ order Butterworth.}$$

– As usual, it is simpler to solve the whole circuit, if that's all we want. However, if there is any variation contemplated in source or load, the z-parameters summarize the action of the circuit, so reduce subsequent work.