SIMON FRASER UNIVERSITY

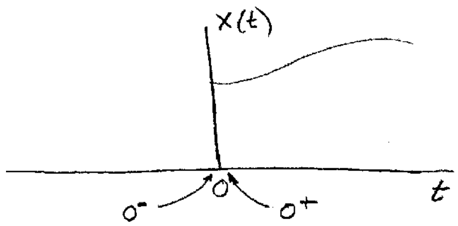
SCHOOL OF ENGINEERING SCIENCE

ENSC 380 LINEAR SYSTEMS

HOW TO GET THE RIGHT INITIAL CONDITIONS WHEN YOU SOLVE DIFFERENTIAL EQUATIONS

1. INTRODUCTION

In systems analysis, you frequently have to solve linear differential equations (DEs) when an input is suddenly applied at time $t = 0$. In such cases, you must distinguish between time $t = 0^-$ (just before any discontinuities in the input) and time $t = 0^+$ (just after the input discontinuities). The sketch below illustrates the situation with a simple step discontinuity in the input $x(t)$.



Most treatments of DE solution run along these lines:

- For $t > 0$, the total solution $y(t)$ is the sum of a particular solution $y_p(t)$ and a homogeneous solution $y_h(t)$, the latter with as many parameters as the order of the system.
- To determine the values of the parameters, fit the total solution to initial conditions (ICs) at time $t = 0^+$. These ICs are typically $y(0^+)$, $\dot{y}(0^+)$, $\ddot{y}(0^+)$ and so on, where the overhead dots denote derivatives.

That's not bad advice – except that you're not usually given ICs at $t = 0^+$. You normally have them at $t = 0^-$, instead. Example: a system initially at rest, in which $y(t)$ and all its derivatives are zero *before* the input is applied. And, unfortunately for you, many of the values change across the transition to $t = 0^+$. So now what?

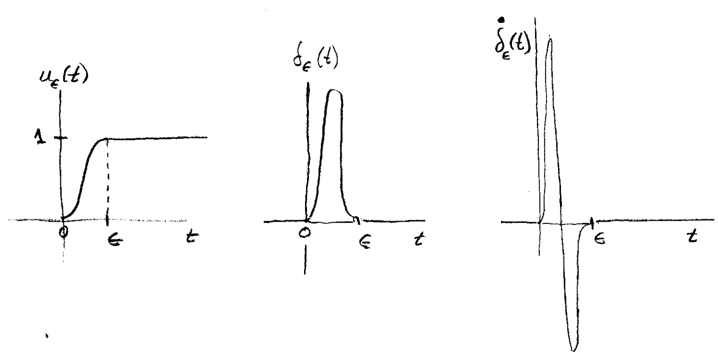
This note shows you how to obtain the ICs at time $t = 0^+$, so that you can then use the standard method from that point. We will assume that the output does not itself contain impulses.

Incidentally, solution of DEs by Laplace transform circumvents this problem. So why not just do that and forget about all the pain in the notes below? One reason is completeness. Generally, you should be able to describe and analyze any system in both the time domain (as here) and the frequency, or transform, domain. Of course, any specific problem may be easier in one domain than the other, but you should be able to do it in either.

2. A DETOUR THROUGH DISCONTINUITIES

In this section, we'll approximate a class of discontinuities (the step, the impulse and their higher derivatives) by differentiable functions, then we'll look at their integrals. This will build a set of mini-results that we can apply in a straightforward way to the IC problem.

Here are the approximate discontinuities (the approximated step and its derivatives):



All of the action in these functions has finite support in t, namely $[0, \epsilon]$, where the intent is to shrink ϵ to zero. Time $t = 0^-$ is just before $t = 0$ and time $t = 0^+$ is just after $t = \epsilon$. Now consider the area under these functions from 0^- to 0^+:

- The area under $u_\epsilon(t)$ is 0, since the function is finite and the interval is vanishingly small.
- The area under $\delta_\epsilon(t)$ is 1, since the approximate step has climbed from 0 to 1.
- The area under $\dot{\delta}_\epsilon(t)$ is 0, since $\delta_\epsilon(t)$ starts and ends with a value of 0. The same argument applies to all of the higher level approximate discontinuities, so they all have area equal to zero.

Having established properties of the discontinuities, we turn to $y(t)$, also assumed to be integrable (and similar properties hold for the input $x(t)$). With a single integration, we have

$$\int_{0^-}^{0^+} y(t)dt = 0$$

$$\int_{0^-}^{0^+} \dot{y}(t)dt = y(0^+) - y(0^-)$$

$$\int_{0^-}^{0^+} \ddot{y}(t)dt = \dot{y}(0^+) - \dot{y}(0^-)$$

and so on. The first integral follows from our assumption that $y(t)$ does not contain impulses.

We can also integrate twice, where

$$\int_{0^-}^{0^+}\int y(t)dt \quad \text{denotes} \quad \int_{0^-}^{0^+}\int^{t} y(a)\,da\,dt;$$

that is, we first obtain the indefinite integral, then integrate it from 0^- to 0^+. Doing so gives

$$\int_{0^-}^{0^+}\int \dot{y}(t)dt = \int_{0^-}^{0^+} y(t)dt = 0$$

$$\int_{0^-}^{0^+}\int \ddot{y}(t)dt = y(0^+) - y(0^-)$$

$$\int_{0^-}^{0^+}\int \dddot{y}(t)dt = \dot{y}(0^+) - \dot{y}(0^-)$$

and so on.

3. BACK TO INITIAL CONDITIONS IN DEs

Having clarified the role of discontinuities at time 0, we can return to the orginal objective: how to find IC values at $t = 0^+$ from those at $t = 0^-$. The easiest way to show it is by an example. Consider a set of three DEs, where the right hand sides increase in complexity

$$\ddot{y}(t) + 4\dot{y}(t) - 6y(t) = 3x(t)$$

$$\ddot{y}(t) + 4\dot{y}(t) - 6y(t) = -5\dot{x}(t) + 3x(t)$$

$$\ddot{y}(t) + 4\dot{y}(t) - 6y(t) = 7\ddot{x}(t) - 5\dot{x}(t) + 3x(t)$$

For each one, we'll obtain the ICs at $t = 0^+$, assuming the sytems to be initially at rest and the input to be proportional to the unit step $x(t) = 3u(t)$.

First DE

Integrate the DE (i.e., all terms) twice over the interval and obtain

$$y(0^+) - y(0^-) + 4\cdot 0 - 6\cdot 0 = 3\cdot 0$$

Therefore $y(0^+) = y(0^-)$. Next, integrate the DE just once over the interval and obtain

$$\left(\dot{y}(0^+) - \dot{y}(0^-)\right) + 4\left(y(0^+) - y(0^-)\right) - 6\cdot 0 = 3\cdot 0$$

or

$$\left(\dot{y}(0^+) - \dot{y}(0^-)\right) + 0 + 0 = 0.$$

Therefore $\dot{y}(0^+) = \dot{y}(0^-)$. That was easy – both of the ICs (*y* and its derivative) at 0^- and 0^+ are the same. Since the sytem was at rest, we summarize by $y(0^+) = 0$ and $\dot{y}(0^+) = 0$.

Second DE

Try the same thing, since it worked before. Integrate the DE twice over the interval to obtain

$$y(0^+) - y(0^-) + 4\cdot 0 - 6\cdot 0 = -5\cdot 0 + 3\cdot 0$$

so $y(0^+) = y(0^-)$. Next, integrate the DE once over the interval to find

$$\left(\dot{y}(0^+) - \dot{y}(0^-)\right) + 4\left(y(0^+) - y(0^-)\right) - 6\cdot 0 = -5\left(x(0^+) - x(0^-)\right) + 3\cdot 0$$

or

$$\left(\dot{y}(0^+) - \dot{y}(0^-)\right) + 0 + 0 = -15.$$

Consequently, $\dot{y}(0^+) = \dot{y}(0^-) - 15$. This demonstrates that the ICs on the derivative are different at 0^- and 0^+. Summary: $y(0^+) = 0$ and $\dot{y}(0^+) = -15$.

Third DE

Finally, we'll tackle the big one, with three terms on the right side. Integrate twice:

$$y(0^+) - y(0^-) + 4\cdot 0 - 6\cdot 0 = 7\left(x(0^+) - x(0^-)\right) - 5\cdot 0 + 3\cdot 0$$

or

$$y(0^+) - y(0^-) = 21$$

This gives us $y(0^+) = y(0^-) + 21$. Next, integrate once:

$$\left(\dot{y}(0^+) - \dot{y}(0^-)\right) + 4\left(y(0^+) - y(0^-)\right) - 6\cdot 0 = 7\cdot 0 - 5\left(x(0^+) - x(0^-)\right) + 3\cdot 0$$

or

$$\left(\dot{y}(0^+) - \dot{y}(0^-)\right) + 84 = -15$$

so that $\dot{y}(0^+) = \dot{y}(0^-) - 99$. Summary: $y(0^+) = 21$ and $\dot{y}(0^+) = -99$.

4. WRAPUP

You’ve seen a tedious, but straightforward, way to get the ICs immediately after discontinuities introduced by switching on the input. However, these notes specifically excluded the possibility of impulses in $y(t)$ itself – and this could happen if you want the impulse response of a system described by a DE in which the degree of the right and left sides of the DE are equal. My advice: calculate the step response instead and differentiate it.