SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 380 Linear Systems

SKETCHING ASYMPTOTIC BODE PLOTS

1. INTRODUCTION

The transfer function of a broad class of systems is conveniently represented by a Bode plot. This is a graph of the magnitude in dB and the phase in degrees, both against frequency on a log scale. Why use dB for the magnitude? Two reasons: first, we are interested in the response over a large dynamic range, usually many orders of magnitude, so the logarithmic scale of decibels is a natural one; and second, the transfer function of a cascade of subsystems is the product of the individual transfer functions (assuming buffering between them to remove loading effects), so that we can add the magnitudes in dB and we can add the phases. But why use a log scale for frequency? Here it's a little more subtle: for systems with rational polynomial transfer functions (i.e., the numerator and denominator are both polynomials in the generalized frequency variable *s*), the asymptotic behaviour is *s* raised to some power, which produces a straight line on log-log paper (dB vs. log frequency).

You may be wondering why you should be concerned with sketching any function, when it is straightforward to plot a transfer function using a computer. Two answers: speed and insight. For transfer functions of low degree, it is a lot faster to do a quick sketch with pencil than to fire up the computer and do a lot of typing (although for high degree polynomials, you'll have to resort to computers if obvious approximations fail). Also, you learn more about the behavior of the system through thinking out a sketch than you do from watching the computer plot it. In this note, you will learn how to make a quick sketch of the Bode plot that captures the essential behaviour.

First, an example. Here's a transfer function

$$T(s) := \frac{4\cdot s + 8}{s^2 + 100\cdot s}$$

and here is a definition of dB:

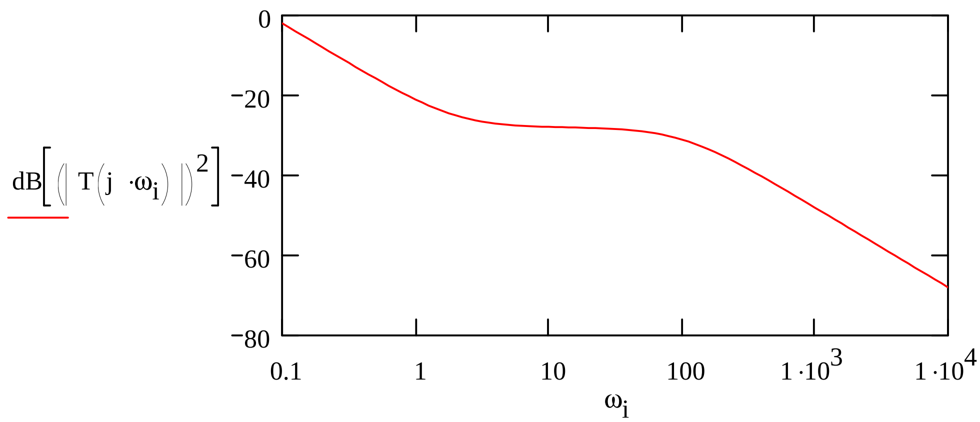
$$dB(x) := 10\cdot \log(x)$$

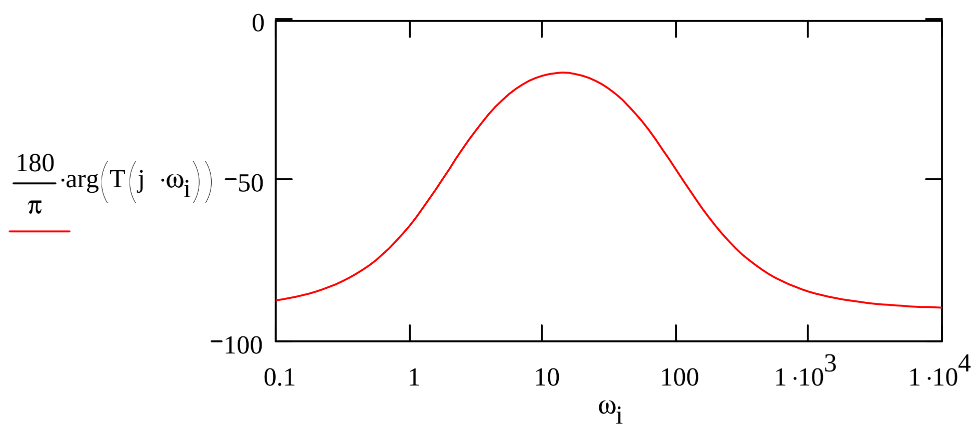
Define a range for the plot by

$$\omega_{inc} := 10^{0.05} \qquad i := 0 .. 100 \qquad \omega_i := 0.1\cdot \omega_{inc}^{\ i}$$

In the plots below, notice that the magnitude has three linear regimes. Less obvious is the fact that the phase curve is the sum of a rising characteristic, followed by a falling one.

We substitute $s=j\omega$ to evaluate the transfer function as a frequency response, and get





2. SKETCHING THE TRANSFER FUNCTION

2.1 Preparing the Transfer Function

The first step in sketching is to factor the transfer function into terms of the form $1+s/a$ where a is the corner frequency (reciprocal time constant). We'll look at quadratic factors with complex roots in a later section. For our example,

$$T(s) = \frac{4 \cdot s + 8}{s^2 + 100 \cdot s} = \frac{8}{100} \cdot \frac{1}{s} \cdot \frac{1 + \frac{s}{2}}{1 + \frac{s}{100}}$$

Roots of the numerator are called zeroes (we have one at s=-2) and roots of the denominator are called poles (we have one at s=0 and one at s=-100). The transfer function is a product (or quotient) of elementary factors, so we produce a sketch for each one, and add (or subtract) them to obtain the plot we want. In the following, you'll see how to make a quick sketch, without manipulating equations, for each of the elementary factors.

2.2 Bode Plots of the Elementary Factors

The Constant Factor

The gain and phase of the constant factor are just

$$G_1 := dB\left[\left(\frac{8}{100}\right)^2\right] \qquad G_1 = -21.938 \qquad \phi_1 := 0$$

You can do this one in your head, since 8/100 is a little less than 1/10, which is -20 dB when squared. Also 10 is 25% more than 8, and 25% is about 1 dB; but it's squared, so we get 2 dB. Hence -20 dB - 2 dB = -22 dB or so.

The Pole at s=0

Moving on to the 1/*s* factor, we have

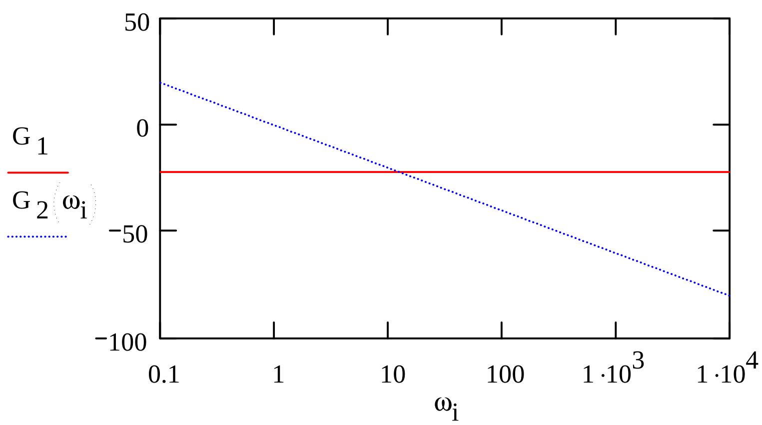
$$G_2(\omega) = dB\left[\left(\left|\frac{1}{j\cdot\omega}\right|\right)^2\right] \qquad \text{or} \qquad G_2(\omega) := -dB\left(\omega^2\right)$$

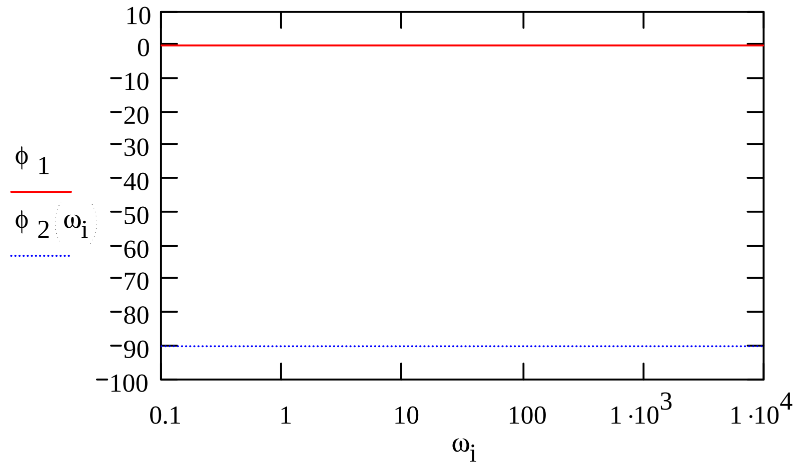
and the phase is just

$$\phi_2(\omega) := -90 \qquad \text{degrees}$$

Here's a critical point: the magnitude equation is there only to generate the plot below - you don't have to remember or use it to produce your sketch. Just note that $G_2(\omega)$ goes through 0 dB at ω=1 and drops at 20 dB per decade increase (factor of 10) of frequency.

The plots below show that the first two factors are easily sketched.





If s had been in the numerator, instead of the denominator, giving a zero at s=0, the sketch would have been equally simple. It would have gone through 0 dB at s=1, as before, but would have *risen* at 20 dB per decade and had a phase of *plus* 90 degrees.

The Zero at s=2

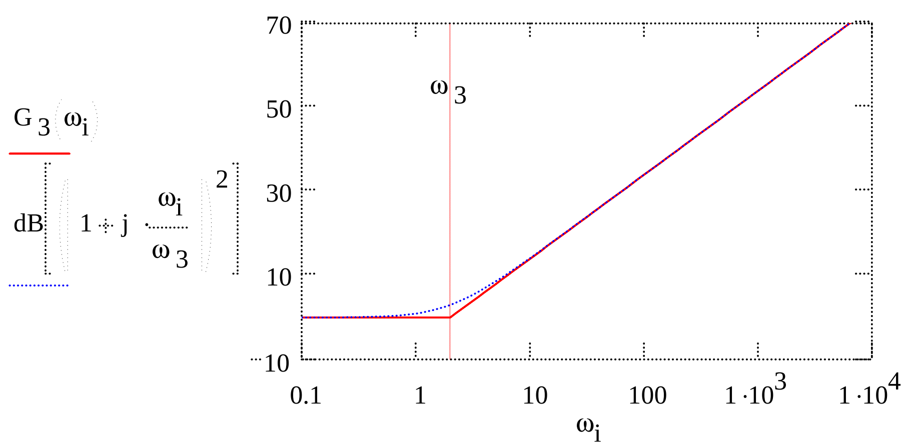
For the remaining numerator and denominator factors, we have to think a little harder. Consider the numerator 1+s/2 first. It has a corner frequency

$$\omega_3 := 2$$

For ω<<2, this factor is asymptotic to 1, and for ω>>2, it's asymptotic to $j\omega/2$. We'll make life very simple, and use only these asymptotes. First, the magnitude:

$$G_3(\omega) := \begin{cases} dB(1) & \text{if } \omega < \omega_3 \\ dB\left[\left(\dfrac{\omega}{\omega_3}\right)^2\right] & \text{otherwise} \end{cases}$$

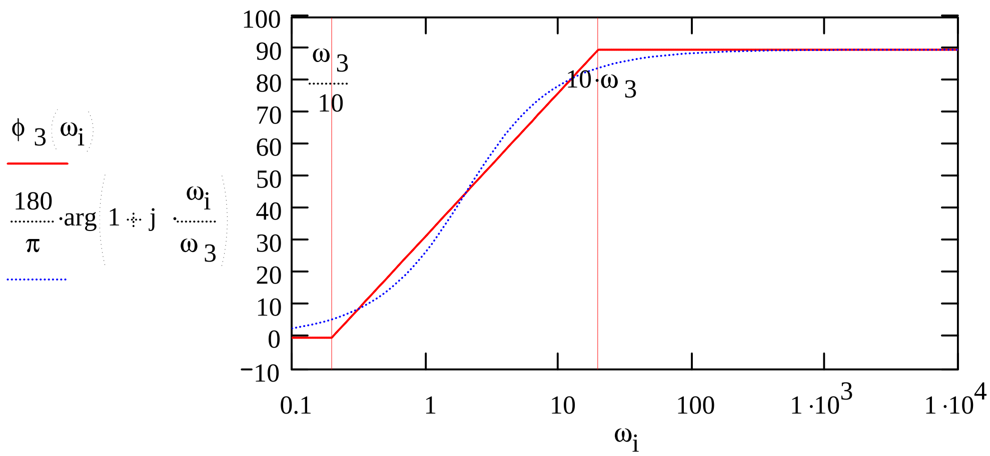
This is very easy - it's constant at 0 dB for ω<2, then rises at 20 dB per decade after that. For additional accuracy, we note that the magnitude of 1+$j\omega$/2 is sqrt(2), or 3 dB, at the corner frequency ω=2. Compare the easily sketched asymptote and the real thing in the plot below. Again, note that you don't have to remember or use the expression for $G_3(\omega)$ - all you have to do is sketch it.



The phase of $1+j\omega/2$ is a little more difficult. For $\omega<<2$, it is asymptotic to 0 degrees, and for $\omega>>2$, it's asymptotic to 90 degrees. What does the transition between asymptotes look like? An easy point is $\omega=2$, at which the phase is 45 degrees. The usual approximation is a straight line transition from 0 degrees at a decade below the corner frequency to 90 degrees at a decade above the corner frequency, passing through 45 degrees at the corner frequency. Mathematically, this is

$$\phi_3(\omega) := \begin{cases} 0 & \text{if } \omega<0.1\cdot\omega_3 \\ 45\cdot\log\left(\dfrac{\omega}{0.1\cdot\omega_3}\right) & \text{if } 0.1\cdot\omega_3\le\omega<10\cdot\omega_3 \\ 90 & \text{otherwise} \end{cases}$$

but you don't have to use this equation. Just sketch it as straight lines.



Remember - a decade above and a decade below the corner frequency.

The Pole at s=100

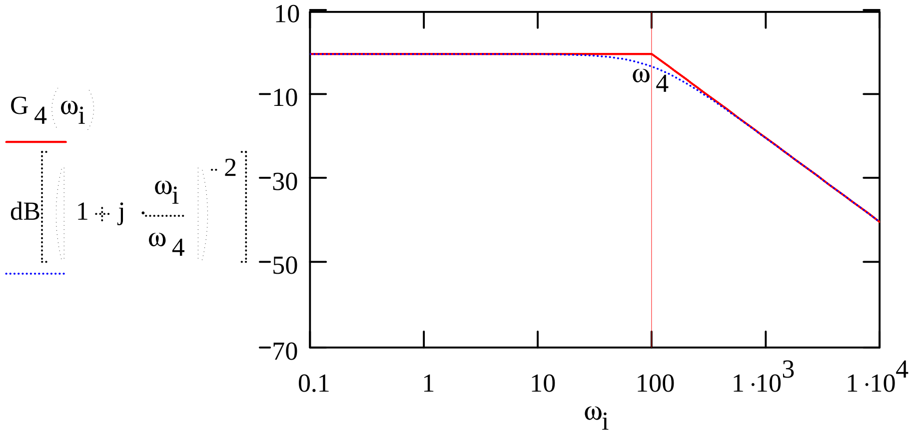
The final factor is the pole $(1+s/100)^{-1}$. It has a corner frequency

$$\omega_4 := 100$$

and is asymptotic to 1 for $\omega<<100$ and asymptotic to $100/j\omega$ for $\omega>>100$. Following the same logic as for the zero above, we have the expression for the gain, using asymptotes only,

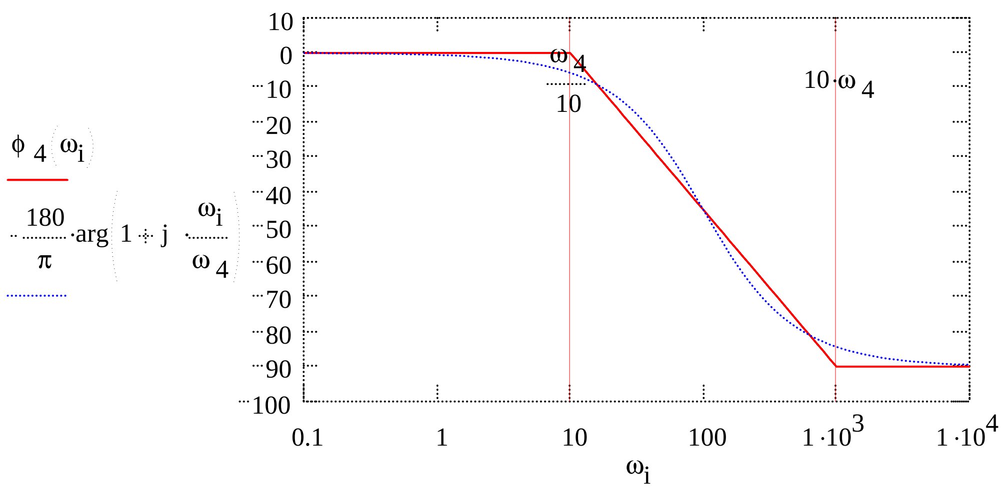
$$G_4(\omega) := \begin{cases} dB(1) & \text{if } \omega<\omega_4 \\ dB\left[\left(\dfrac{\omega_4}{\omega}\right)^2\right] & \text{otherwise} \end{cases}$$

In sketch form, it's constant at 0 dB up to $\omega=100$, then drops at 20 dB per decade. At the corner frequency $\omega_4=100$, it's 1/sqrt(2), or -3 dB. Compare this approximation and the real thing in the plot below.



The phase for the pole is also similar to that for the zero, except that it goes negative.

$$\phi_4(\omega) := \begin{cases} 0 & \text{if } \omega < 0.1\cdot\omega_4 \\ -45\cdot\log\left(\dfrac{\omega}{0.1\cdot\omega_4}\right) & \text{if } 0.1\cdot\omega_4 \le \omega < 10\cdot\omega_4 \\ -90 & \text{otherwise} \end{cases}$$



2.3 Putting It Together

Now that you know how to sketch the individual factors, we can summarize the method as follows:

* Factor the transfer function.

* For each factor, sketch its asymptotic Bode plot, putting all plots on the same graph. A squared pad gives good enough accuracy.

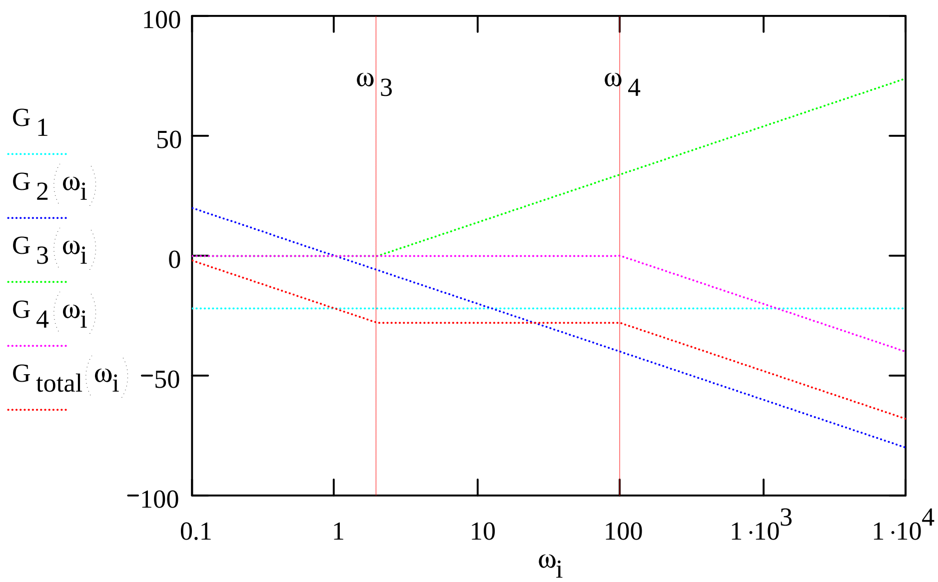
* Add the asymptotic Bode plots (because both dB magnitudes and phases add in complex multiplication) and sketch the sum.

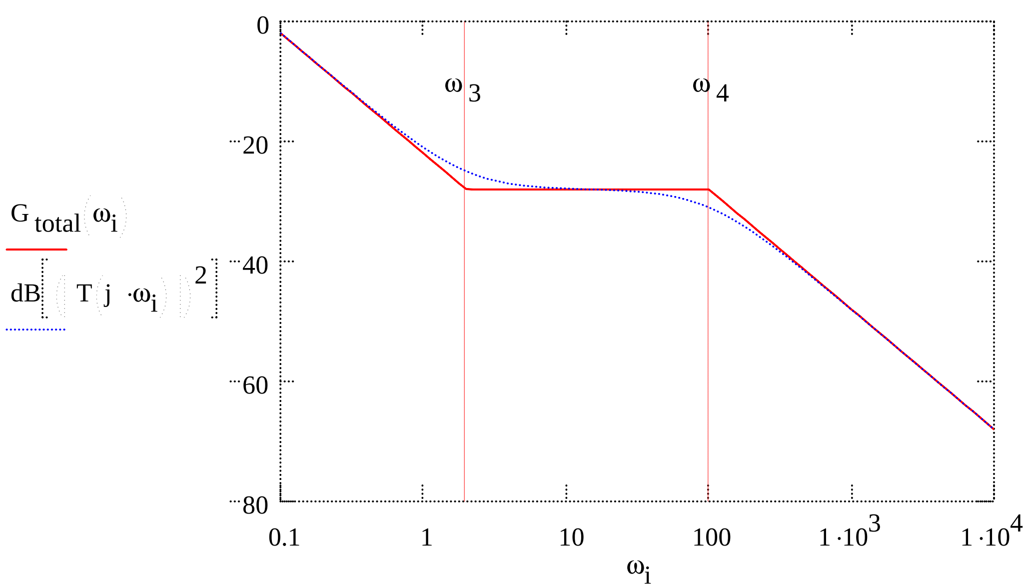
* Round the corners a little (3 dB for magnitudes and go through the 45 degree point for phases).

In the case of repeated factors, such as $(1+s/2)^2$, you can treat them as separate factors, or simply double the change of magnitude or phase.

The plots below show, first, the relation of the asymptotic sketches of the sum and the individual factors and, second, the relation of the sum to the true plot we saw on the second page.

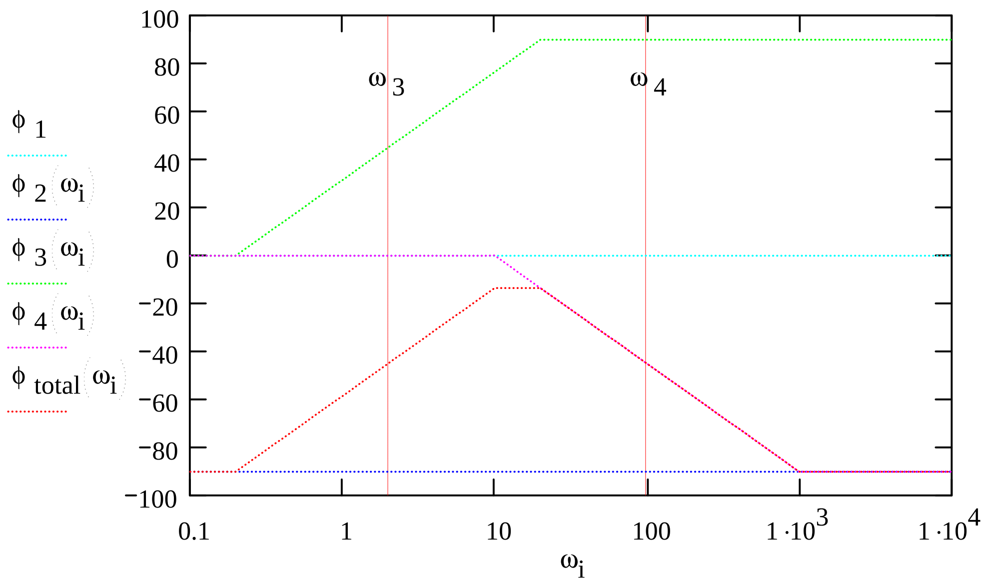
$$G_{total}(\omega) := G_1 + G_2(\omega) + G_3(\omega) + G_4(\omega)$$

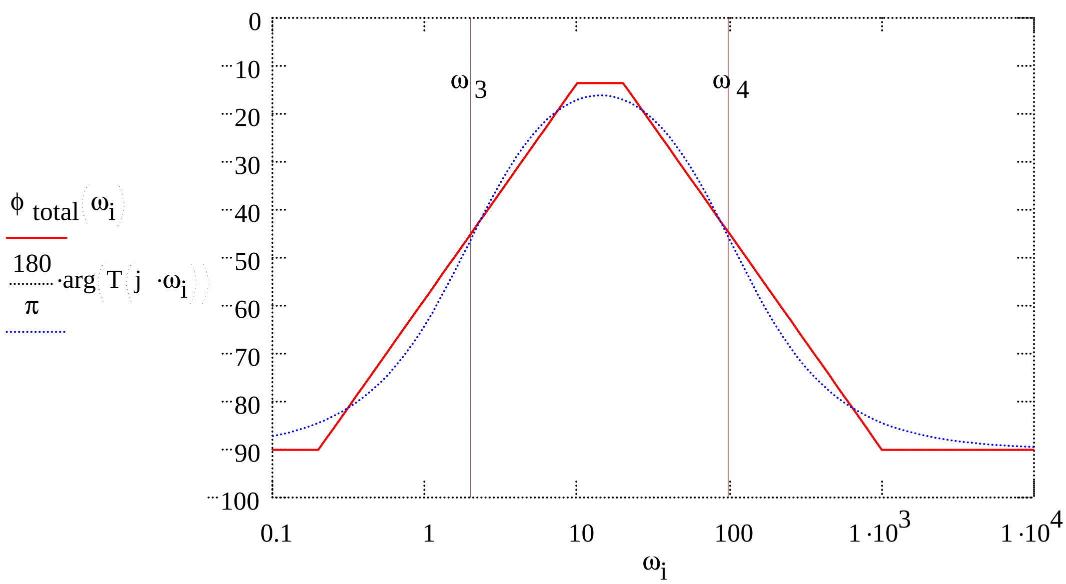




and for the phase:

$$\phi_{total}(\omega) := \phi_1 + \phi_2(\omega) + \phi_3(\omega) + \phi_4(\omega)$$





Not bad accuracy for an easy sketch method!