1.7 Linearity — The Big One

- Almost all real world systems are non linear. However, a special class of system — the linear system — is widely (almost exclusively, at times) used as a model in science, engineering, economics, etc.

  Why? — simple structure allows complete analytical solutions
  — captures dynamic effects
  — a reasonable model for many almost-linear systems, perhaps over a restricted range of values.

- Definition: $\xrightarrow{x(t)} \boxed{\text{system}} \xrightarrow{y(t)}$

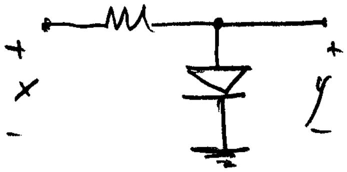
  — A system is linear iff the response to
  $$x(t) = x_1(t) + x_2(t)$$
  is $$y(t) = y_1(t) + y_2(t)$$
  where $x_1(t)$ produces $y_1(t)$
  $x_2(t)$ produces $y_2(t)$ for any choice of $x_1(t)$, $x_2(t)$

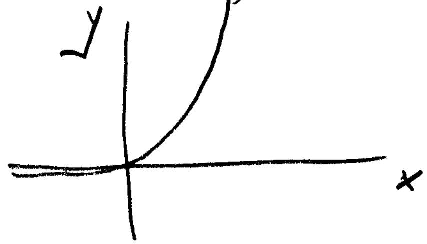
  — Special case: if $x(t)$ produces $y(t)$
  then $\alpha x(t)$ produces $\alpha y(t)$

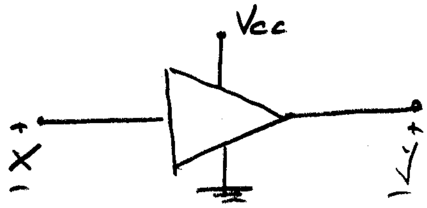
  — or $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ produces $\alpha_1 y_1(t) + \alpha_2 y_2(t)$

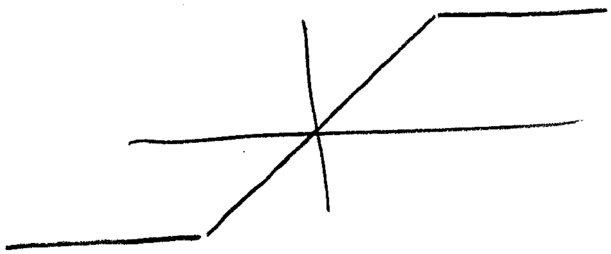
- If we can represent $x(t) = \sum_{i=1}^{N} \alpha_i \varphi_i(t)$ and we know the response $\eta_i(t)$ to $\varphi_i(t)$, then
  $$y(t) = \sum_{i=1}^{N} \alpha_i \eta_i(t)$$
  for any choice of $\{\alpha_1, \cdots \alpha_N\}$

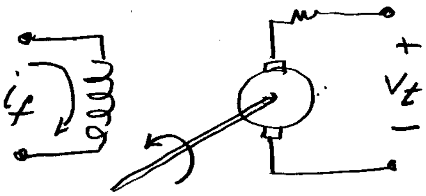
- Memoryless nonlinearities are familiar:

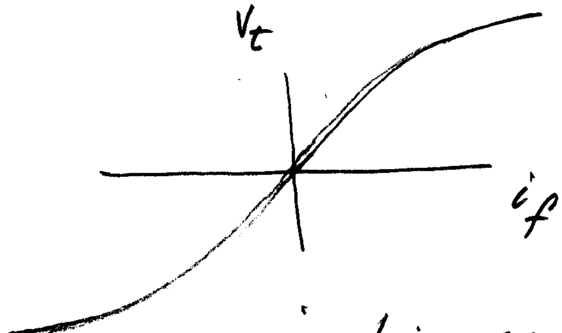






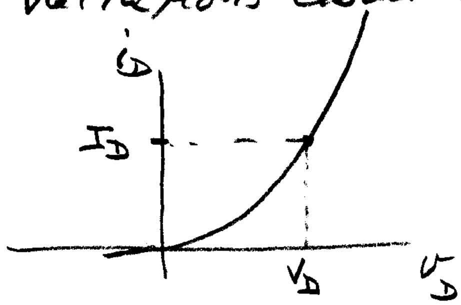






The lack of memory is an approximation, as is evident from the slight hysteresis.

- Some systems can be linearized by considering small variations about some bias point.

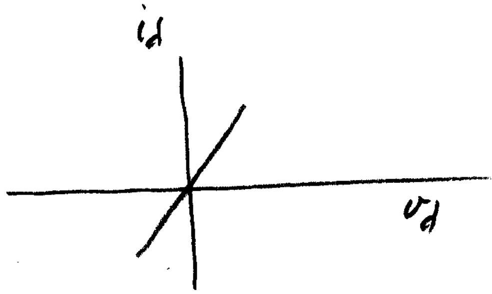


$$v_D = V_D + v_d$$

$$i_D = I_D + i_d$$

The incremental gain depends on the bias point.

This is just a Taylor series.



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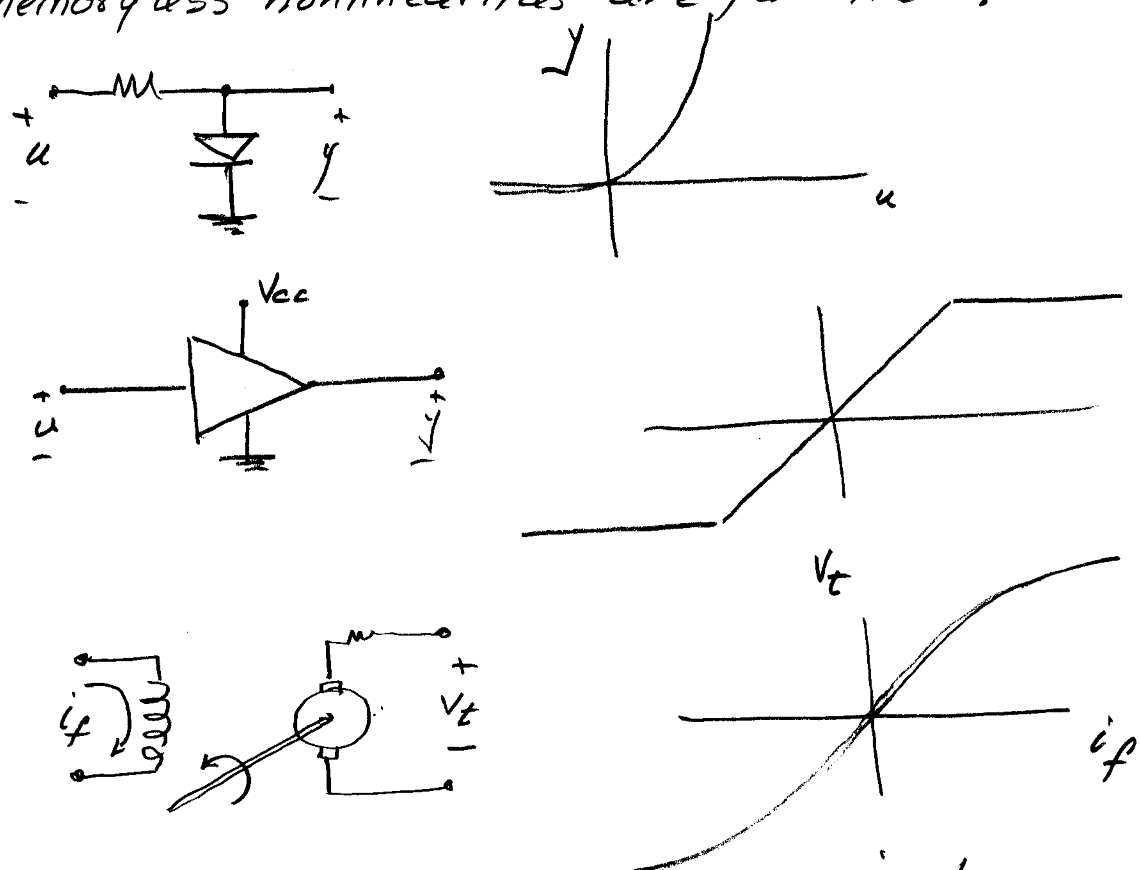

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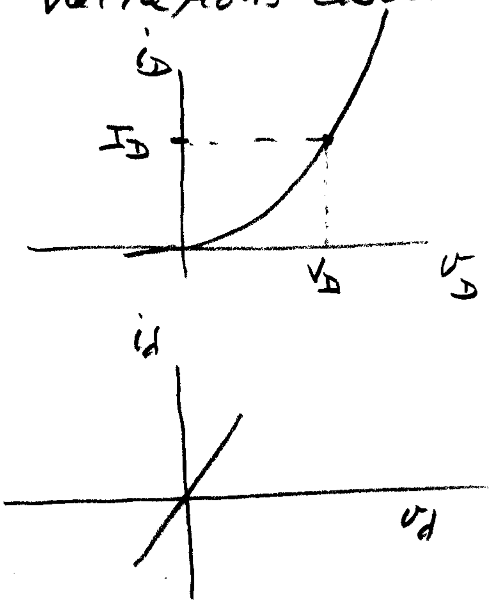

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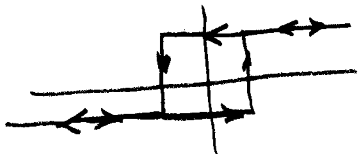
Some systems cannot be linearized:



this has memory

Why linearize? – simpler analysis, more insight
– sufficient accuracy for the purpose.

- Nonlinear dynamic systems are capable of truly weird behaviour. One example:

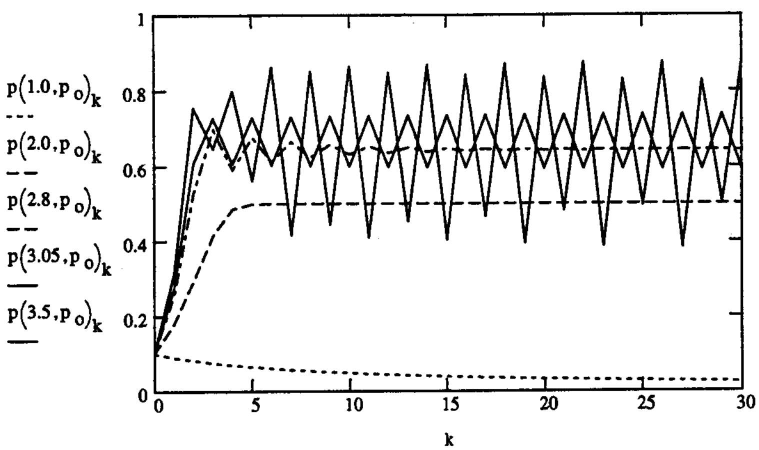
THE LOGISTIC EQUATION: $p_i = r\, p_{i-1}\,(1-p_{i-1})$

This is a very simple yet famous nonlinear dynamic system.

$N := 30 \qquad k := 0..N \qquad p_0 := 0.1$

$$p(r, init) := \left|\begin{array}{l} p_0 \leftarrow init \\ \text{for } i \in 1..N \\ \quad p_i \leftarrow r \cdot p_{i-1} \cdot (1 - p_{i-1}) \\ p \end{array}\right.$$

Parameter *r* represents rate of population growth. Factor $(1-p_{i-1})$ represents inhibiting effect of overpopulation.



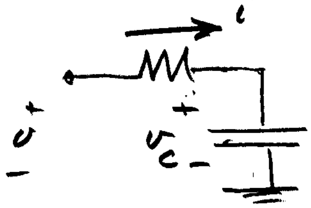
r=1.0, population collapses
r=2.0, population grows to limit
r=2.8, overshoot, settles to steady population
r=3.05, oscillation between two values
r=3.5, oscillation between four values

Other strange behaviour includes chaos – extreme sensitivity to initial conditions, so that even minute perturbations lead to major changes in long term response.

- In contrast, the output of a linear dynamic system is a <u>linear combination of</u> all present and past input values (and future ones, if non causal).

1.8 The Concept of State

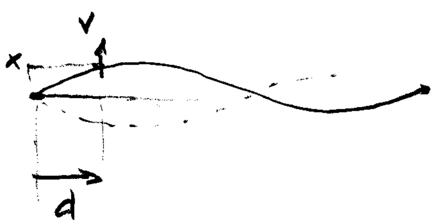
- The state of a system captures all past history, so that the output depends on current input and current state.



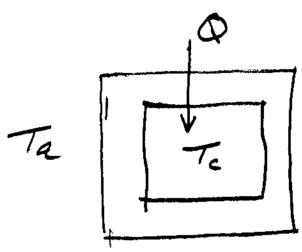
The cap voltage summarizes all prior variations of $v(t)$.

If $i(t)$ is output

$$i(t) = \frac{v(t) - v_c(t)}{R}$$

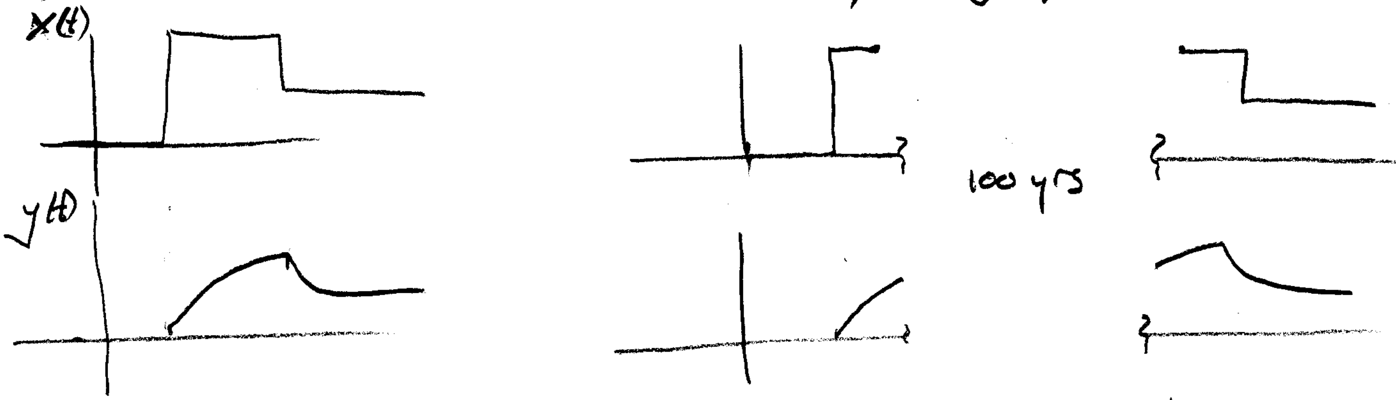


The state of the vibrating string is the displacement and velocity of every point.



The temperature in the chamber is the state. The heat flow rate Q depends on the input (ambient temp) and the state.

- If we could save, then restore the state, the system would not be affected by a gap.



Here the state might be charge on a cap, but a similar concept applies to multitasking on a computer. The state is the set of values in all CPU registers, stack pointers, and data areas of a program. Save them, then restore at some future date and the program simply carries on.

- If the state is non-zero, then the output will be non-zero even if there's no input. This is the "zero-input response".

- If the system is "at rest" – i.e. the state is zero – the output when we apply the input is the "zero state response".