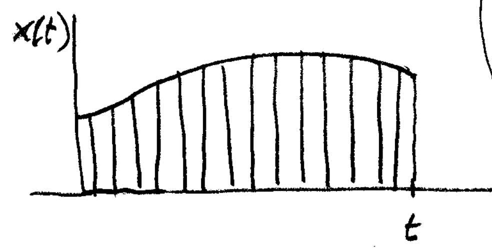
2.4 Arbitrary Inputs – Convolution Integral

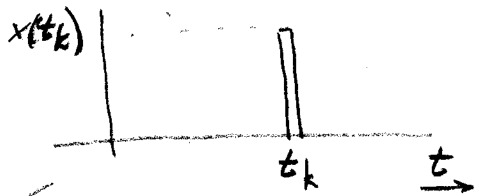
read H & vV
rest of 2.2
then 2.3

Given $h(t)$ and input $x(t)$
What is output at time t, $y(t)$?

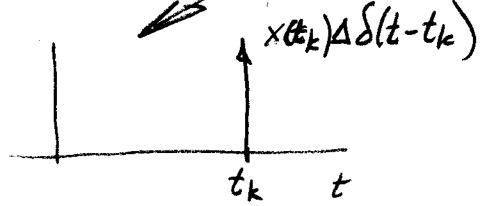


Approx $x(t)$ as staircase, strips Δ wide:
$t_k = k\Delta$, area of strip $\approx x(t_k)\Delta$

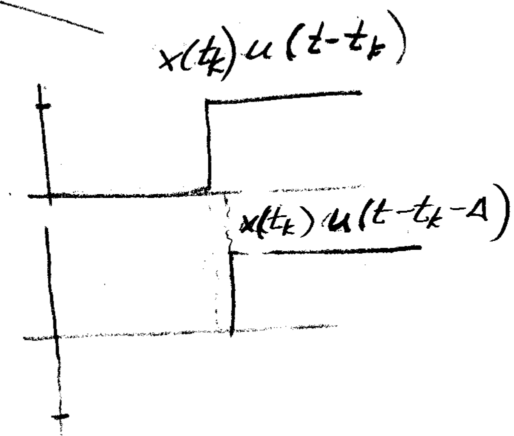
Use superposition, get response to strip k alone



$\approx$



$=$



produces
$x(t_k)\Delta\, h(t-t_k)$

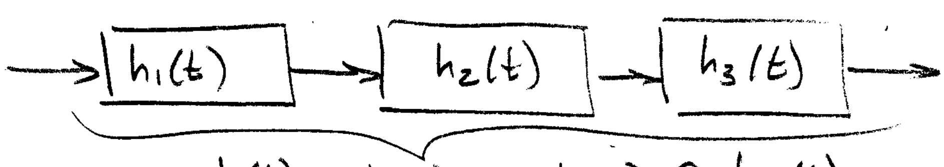
produces
$x(t_k)\left(g(t-t_k)-g(t-t_k-\Delta)\right) \approx x(t_k)h(t-t_k)\Delta$

$$y(t) = \sum_k x(t_k)\, h(t-t_k)\Delta \xrightarrow{\Delta\to 0} y(t) = \int_0^t x(\alpha)\, h(t-\alpha)\, d\alpha$$

$$= x(t) \circledast h(t)$$

"flip, slide, integrate"

cascaded systems

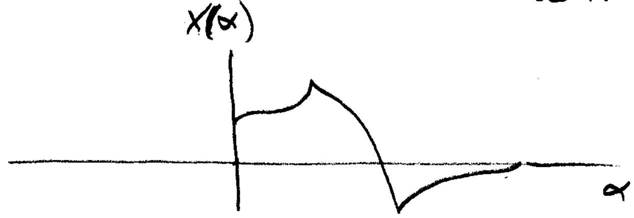


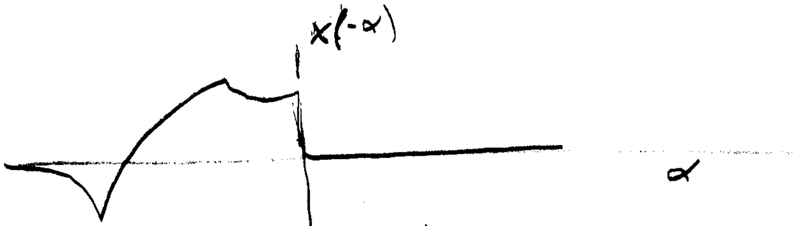
$$h(t) = h_1(t) \circledast h_2(t) \circledast h_3(t)$$

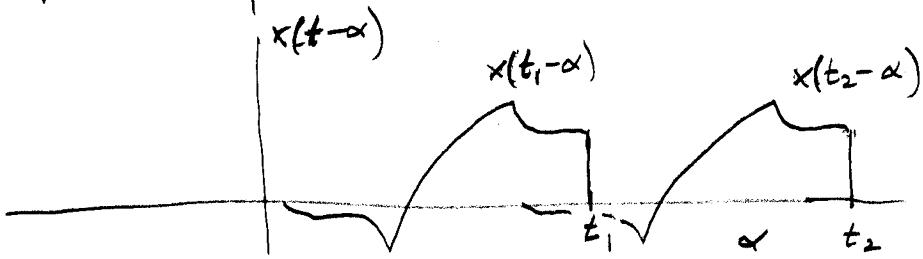
"Flip, slide and integrate"

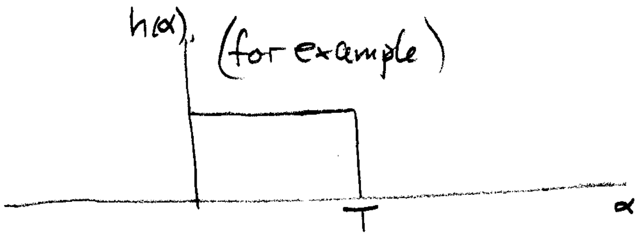
$$y(t) = \int_0^t h(\alpha)\, \underbrace{x(t-\alpha)}\, d\alpha$$

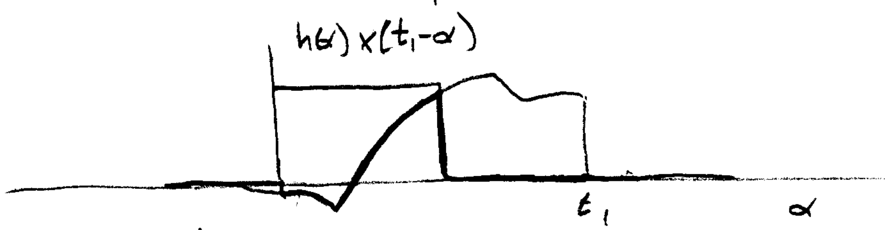
sketch this as a function of α, as it appears in the integrand



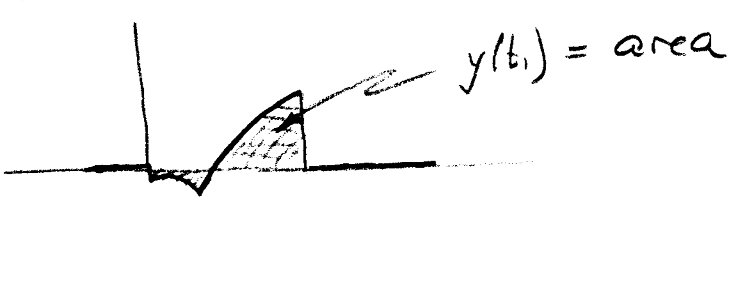




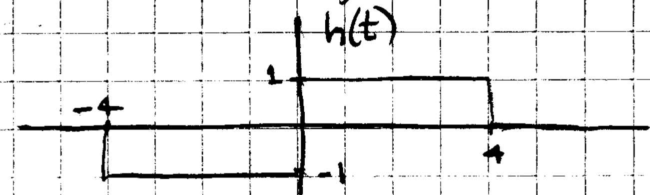


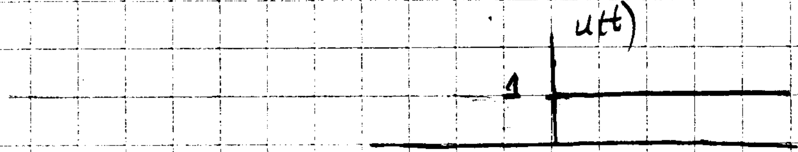


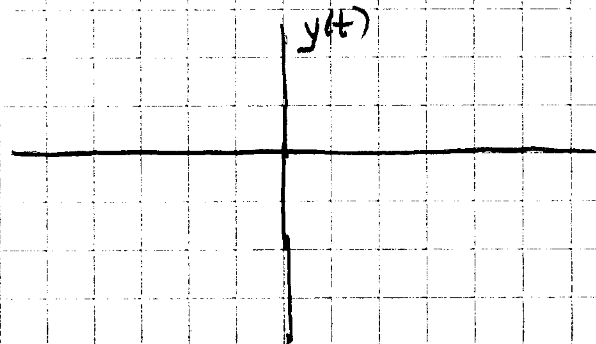
$$y(t_1) = \int_0^{t_1} h(\alpha)\, x(t_1-\alpha)\, d\alpha$$



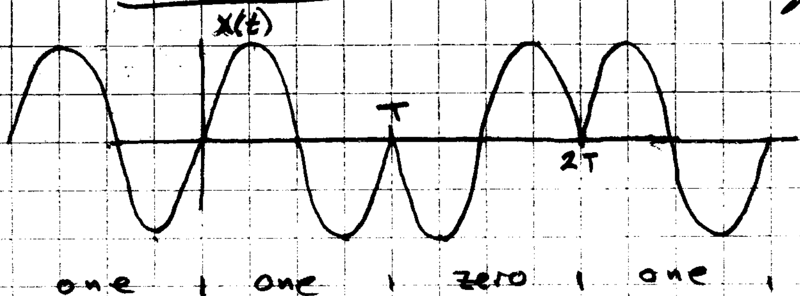
Example Step response of non causal, FIR

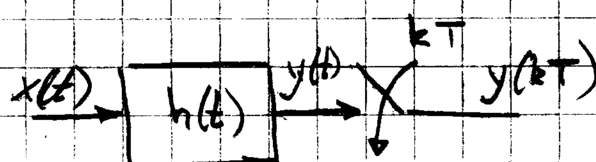


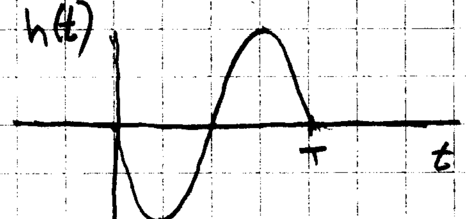




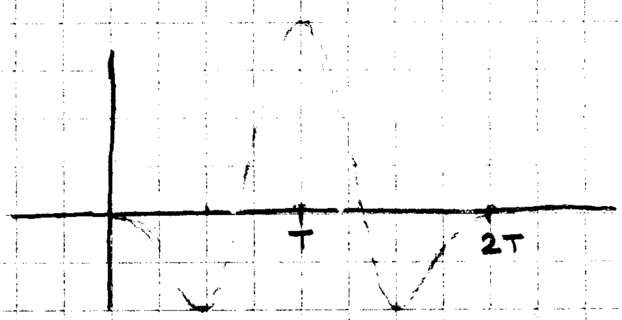
Example Modulated data sequence "binary phase shift keying"







Receive modem response to a single positive pulse

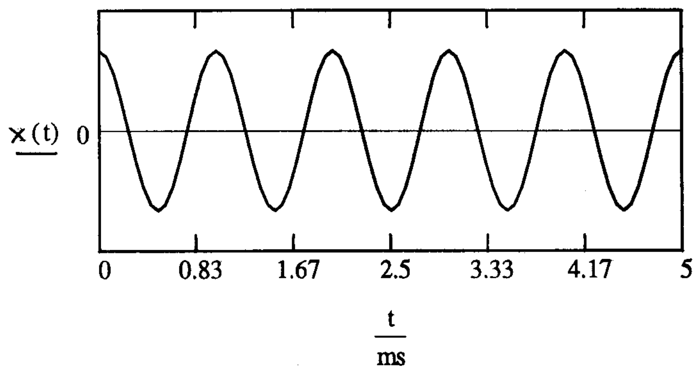


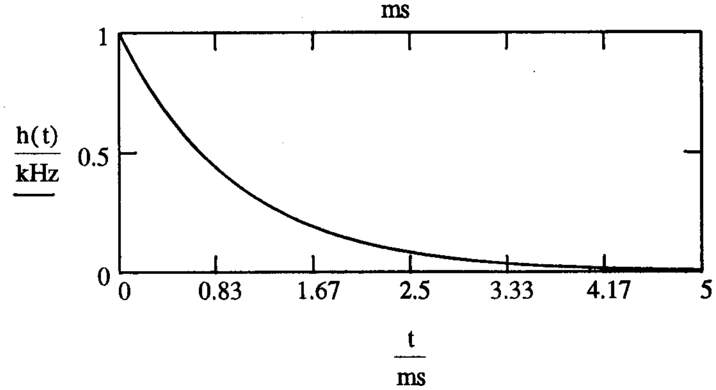
[Website: examples of convolution in continuous time]

Example: First order lowpass system, input is a cosine switched on at time zero. Find the response by convolution - "flip, slide and integrate"

$$f_c := 1 \cdot \mathrm{kHz} \qquad x(t) := \cos\left(2 \cdot \pi \cdot f_c \cdot t\right) \qquad \tau := 1 \cdot \mathrm{ms} \qquad h(t) := \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}}$$

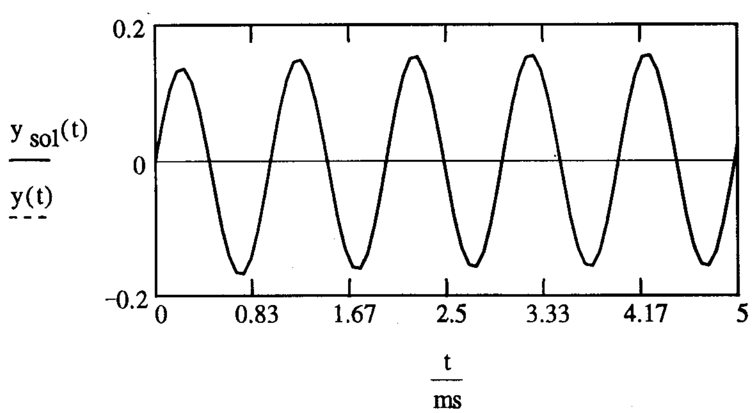
$$t := 0 \cdot \mathrm{ms}, \frac{1}{16 \cdot f_c} .. 5 \cdot \tau$$





$$y(t) := \int_{0 \cdot \mathrm{ms}}^{t} x(\alpha) \cdot h(t - \alpha)\, d\alpha$$

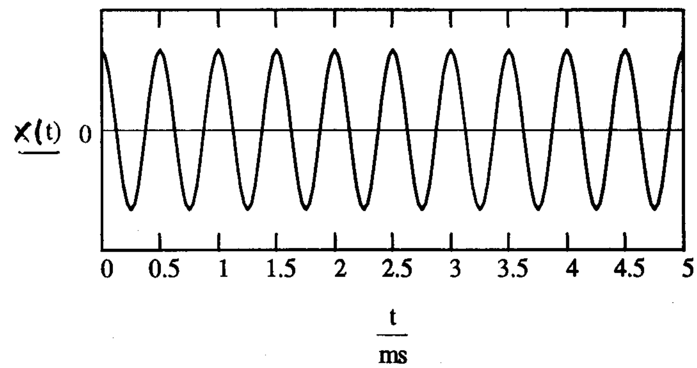
$$y_{sol}(t) := \frac{1}{1 + \tau^2 \cdot \left(2 \cdot \pi \cdot f_c\right)^2} \cdot \left(\cos\left(2 \cdot \pi \cdot f_c \cdot t\right) + 2 \cdot \pi \cdot f_c \cdot \tau \cdot \sin\left(2 \cdot \pi \cdot f_c \cdot t\right) - \exp\left(-\frac{t}{\tau}\right)\right)$$

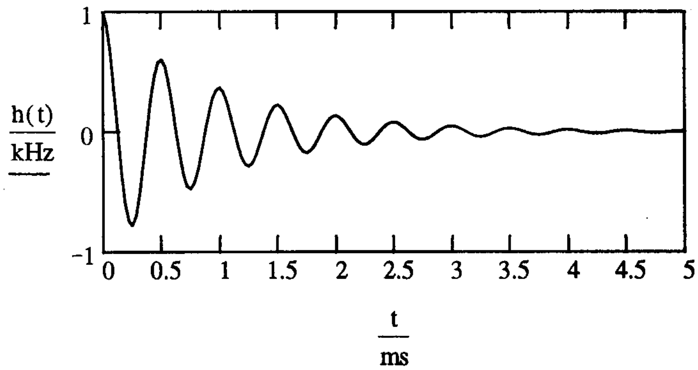


Example: sine wave input to a filter tuned to the same frequency

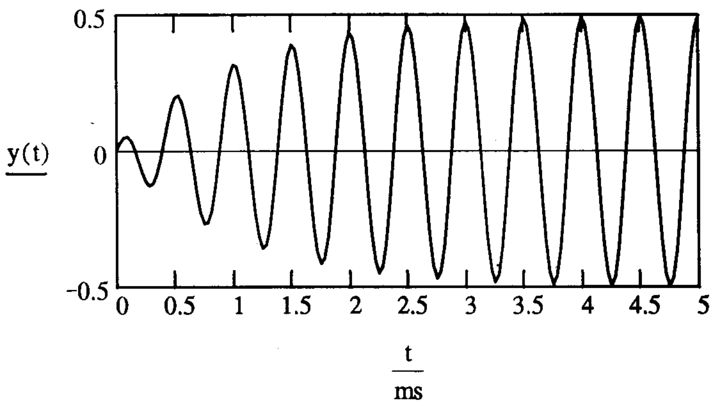
$$f_c := 2\cdot\mathrm{kHz} \qquad x(t) := \cos\left(2\cdot\pi\cdot f_c\cdot t\right) \qquad \tau := 1\cdot\mathrm{ms} \qquad h(t) := \frac{1}{\tau}\cdot e^{-\frac{t}{\tau}}\cdot\cos\left(2\cdot\pi\cdot f_c\cdot t\right)$$

$$t := 0\cdot\mathrm{ms}, \frac{1}{16\cdot f_c} .. 5\cdot\tau$$





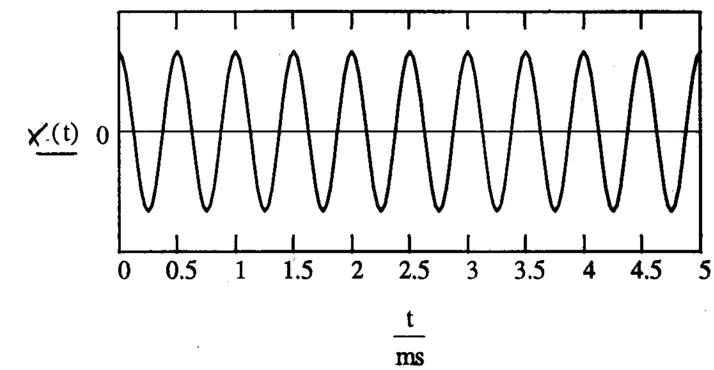
$$y(t) := \int_{0\cdot\mathrm{ms}}^{t} x(\alpha)\cdot h(t-\alpha)\, d\alpha$$

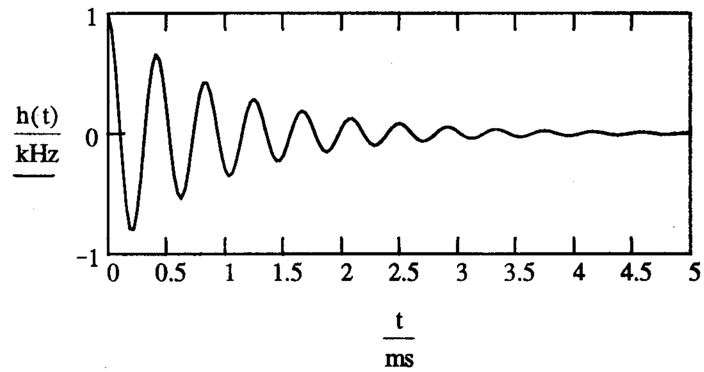


<u>Example:</u> sine wave input to a filter tuned to a slightly different frequency

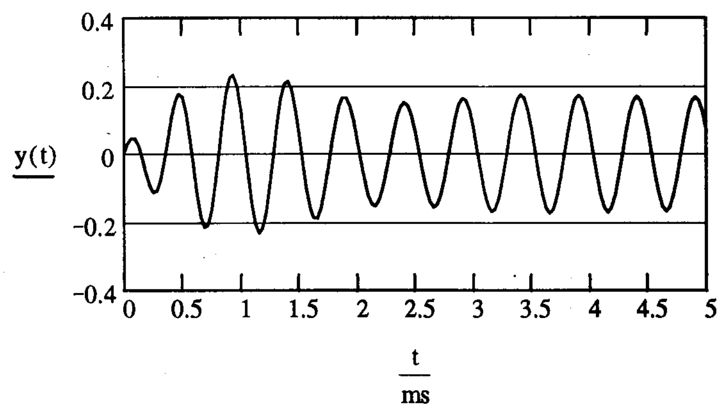
$$f_c := 2\cdot\text{kHz} \qquad x(t) := \cos\left(2\cdot\pi\cdot f_c\cdot t\right) \qquad \tau := 1\cdot\text{ms} \qquad h(t) := \frac{1}{\tau}\cdot e^{-\frac{t}{\tau}}\cdot\cos\left(2\cdot\pi\cdot 1.2\cdot f_c\cdot t\right)$$

$$t := 0\cdot\text{ms}, \frac{1}{16\cdot f_c} .. 5\cdot\tau$$





$$y(t) := \int_{0\cdot\text{ms}}^{t} x(\alpha)\cdot h(t-\alpha)\, d\alpha$$



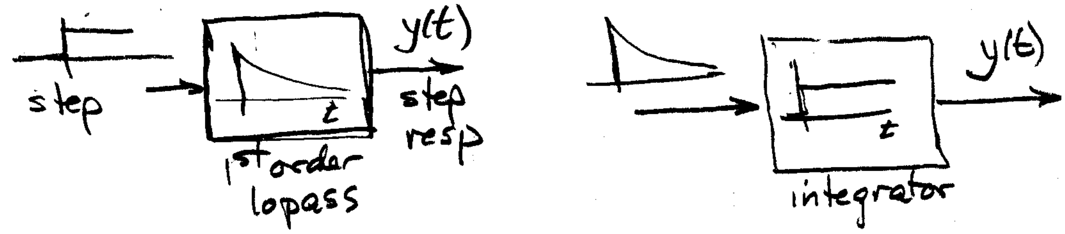
- Other properties
  - commutative
  - associative

<u>example</u>

so

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{\;y(t)\;} \qquad h(t) \longrightarrow \boxed{x(t)} \xrightarrow{\;y(t)\;}$$

eg



<u>example</u>

$$x(t) \longrightarrow \underbrace{\boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_3(t)}}_{h(t) = h_1(t) \circledast h_2(t) \circledast h_3(t)} \xrightarrow{\;y(t)\;}$$

example

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \xrightarrow{\;y(t)\;}$$

$$h_1(t) \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{x(t)} \xrightarrow{\;y(t)\;}$$