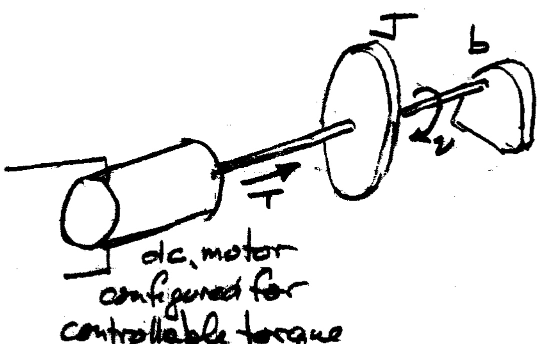
3. FINITE ORDER SYSTEMS IN THE TIME DOMAIN

- So far, we have considered general LTI systems, no constraints on functional form of $h(t)$.
- An important class of LTI is "finite order" systems, ones described by finite degree linear DE or ΔE. Their frequency responses (transfer functions) are rational polynomials

$\boxed{\text{read H \& vV 2.4}}$

3.1 A Few LTI Physical Systems

- A shaft with inertia and viscous friction:



viscous friction $T_f = b\,\nu$

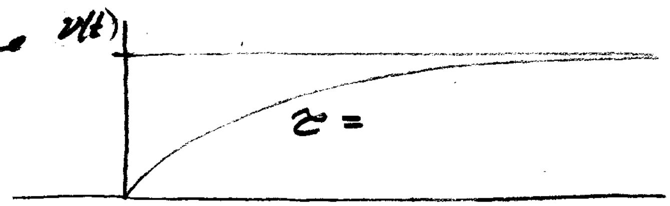
T_f : nt·m ; ν: rad/s

moment of inertia J: kg·m²

torque balance $\quad T - b\nu = J\dot{\nu}$

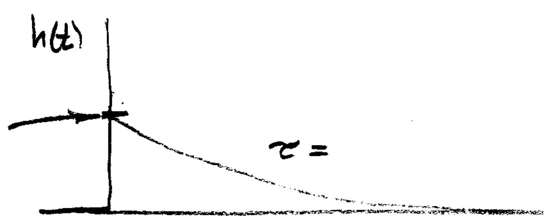
$$\dot{\nu} + \frac{b}{J}\nu = \frac{T}{J} \qquad 1^{st} \text{ order lowpass}$$

step response

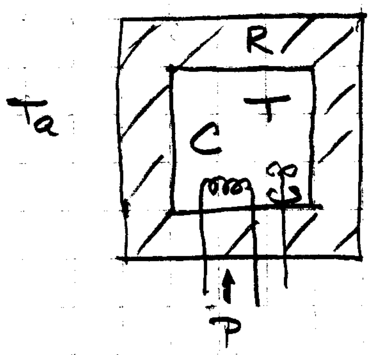


$\nu(\infty) =$

impulse resp



- A thermal chamber



fan evens out heat distribution.

thermal energy in chamber $\quad Q = C\,T$

$\quad C$ heat capacity $\quad J/K$

heat flow rate through walls $\quad \dot{Q} = \dfrac{\Delta T}{R}$

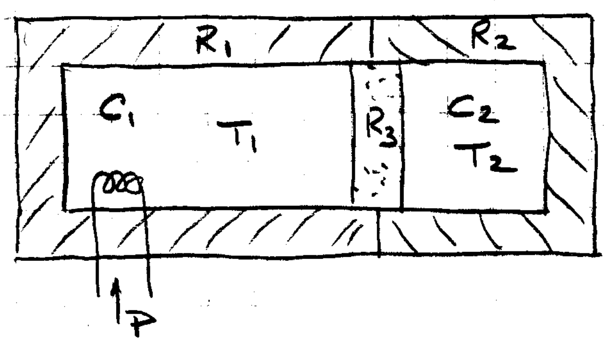
$\quad R$ thermal resistance $\quad K/W$

if power to heater is P watt, then heat flow balance

$$\dot{Q} = C\dot{T} = P - \frac{T - T_a}{R}$$

$$\dot{T} + \frac{1}{RC}T = \frac{P}{C} + \frac{T_a}{RC}$$ 1^{st} order, two inputs

Double chamber:

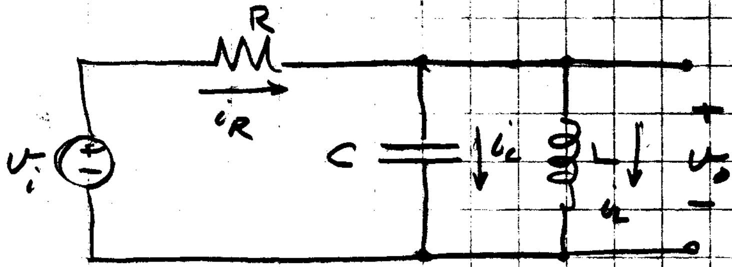


$$C\dot{T}_2 = \frac{T_1 - T_2}{R_3} - \frac{T_2 - T_a}{R_2}$$

$$C\dot{T}_1 = P - \frac{T_1 - T_2}{R_3} - \frac{T_1 - T_a}{R_1}$$

coupled 1^{st} order equations

• A resonant RLC circuit



$$i_c = C\dot{v}_o \qquad i_L = \frac{1}{L}\int v_o\,dt \qquad i_R = i_L + i_c$$

$$v_o = v_i - i_R R$$

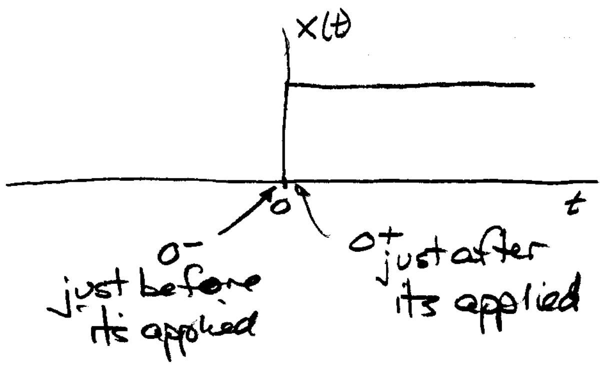
$$v_o = v_i - R\left(C\dot{v}_o + \frac{1}{L}\int_{-\infty}^{t} v_o(\alpha)\,d\alpha\right)$$

$$\dot{v}_o = \dot{v}_i - RC\ddot{v}_o - \frac{R}{L}v_o$$

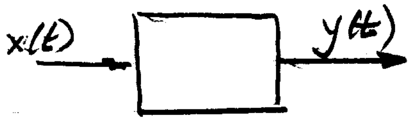
$$\ddot{v}_o + \frac{1}{RC}\dot{v}_o + \frac{1}{LC}v_o = \frac{\dot{v}_i}{RC}$$

3.2 Differential Equations (Review)

• We'll consider inputs that are switched on at time $t=0$



This notation lets us separate initial conditions ($t=0^-$) from conditions immediately after input starts ($t=0^+$). Lets us account for discontinuities.



- $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = b_m x^{(m)} + \cdots b_0 x \quad (m \le n)$

  $y^{(k)}(0-) = c_k \quad$ (ICs)

  homogeneous eq'n has RHS zero.

- zero input response: homogeneous solution with parameters selected to satisfy ICs
- zero state response: homogeneous plus particular with params selected for zero ICs.

- total solution $y_T(t) = y_{zs}(t) + y_{zi}(t)$

  Proof: $A[y_T] = A[y_{zs} + y_{zi}] = A[y_{zs}] + 0 = B[x]\,, \quad t > 0$

  and $y_T^{(k)}(0-) = y_{zs}^{(k)}(0-) + y_{zi}^{(k)}(0-) = 0 + c_k = c_k\,, \quad t = 0^-$

- impulse response – how to calculate it?
  - solve directly with $x(t) = \delta(t)$ using tricky arguments regarding continuity of derivatives
  - or get step response and differentiate.

- Homogeneous solution depends only on system internal dynamics, not on input. It must decay with time for a stable system.

- Obtain homogeneous solution by substituting trial solution e^{st} into homogeneous eq'n:

$$A[e^{st}] = 0$$

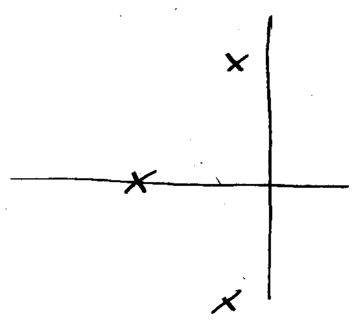
$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) e^{st} = 0$$

 - only certain values work: roots of $A(s) = 0$
 - coeffs are real, so roots are real or conj pairs $a \pm jb$
 - so sol'ns are of form

$$e^{at}$$

$$e^{jbt},\ e^{-jbt} \longrightarrow R\cos(bt+\varphi) \text{ or } A\cos(bt) + B\sin(bt)$$

$$e^{(a+jb)t},\ e^{(a-jb)t} \longrightarrow Re^{at}\cos(bt+\varphi), \text{ etc}$$



real part of roots must be <0 for stable (BIBO) system

 - repeated roots give solution of form

$$\text{poly}(t)\, e^{(a \pm jb)t}$$

- impulse response $h(t)$ is a homogeneous solution with a specific choice of parameters
- a characteristic of finite order systems is that $h(t)$ is a sum of responses of exponential form.
- you can even obtain DE from the impulse response (hence from step resp)