

3.3 Difference Equations

- differences are analogous to derivatives:

sequence	$y(k)$	$y(0)$	$y(1)$	$y(2)$	$y(3)$
first (backward) difference $\Delta^{(1)}y(k)$:		$y(0) - y(-1)$	$y(1) - y(0)$	$y(2) - y(1)$	$y(3) - y(2)$
second (backward) diff $\Delta^{(2)}y(k)$:		$y(0) - 2y(-1) + y(-2)$	$y(1) - 2y(0) + y(-1)$	$y(2) - 2y(1) + y(0)$	$y(3) - 2y(2) + y(1)$
$\Delta^{(3)}y(k)$:		$y(0) - 3y(-1) + 3y(-2) - y(-3)$	$y(1) - 3y(0) + 3y(-1) - y(-2)$	$y(2) - 3y(1) + 3y(0) - y(-1)$	$y(3) - 3y(2) + 3y(1) - y(0)$

- also by analogy ΔE 's e.g. $3\Delta^{(3)}y - \Delta^{(2)}y + 2\Delta y - y = \Delta x - x$
more common to use samples themselves, not explicit differences

$$a_n y(k+n) + a_{n-1} y(k+n-1) + \dots + a_1 y(k+1) + a_0 y(k) \\ = b_m x(k+m) + \dots + b_0 x(k); \quad k \geq -n$$

with IC's: $y(-k) = C_k \quad k = 1 \dots n$

- can change variables (time shift) to an equation that is easier to solve numerically. With $i = k+n$

$$a_n y(i) + a_{n-1} y(i-1) + \dots + a_1 y(i-n+1) + a_0 y(i-n) \\ = b_m x(i+m-n) + \dots + b_0 x(i-n); \quad i \geq 0$$

IC's $y(-i) = C_i \quad i = 1 \dots n$

example $y(k+1) - 2y(k) = x(k+1) + x(k); \quad k \geq -1$

with $y(-1) = 1$

and $x(k) = \begin{cases} 2, & 3, & 3, & 2, & 1, & 0, & 0, & \dots \\ 0 & & & & & & & \end{cases} \quad \begin{matrix} k=0, 1, 2, \dots \\ k < 0 \end{matrix}$

rewrite ΔE $y(i) - 2y(i-1) = x(i) + x(i-1); i \geq 0$ B.3.2
and solve for "next i "

$$y(i) = 2y(i-1) + x(i) + x(i-1) \quad i \geq 0$$

i	y	x
-1	1	0
0		2
1		3
2		3
3		2
4		1

a "template", weights

2	1
	1

example pay off a loan of \$5000.00, equivalent monthly
interest rate 1%, monthly payment \$100.
 $B(k)$ is balance after k payments
 $B(k) = 1.01 B(k-1) - 100, \quad B(0) = 5000$



$$B(k) - (1+r)B(k-1) = -m(k)$$

- As with DE's, $y_T(k) = y_{zs}(k) + y_{zir}(k)$
- pulse response is a homogeneous solution with specific choice of parameters to satisfy RHS with $x(k) = \delta(k)$
- homogeneous solution:
 - numerically iterate $a_n y(k) + a_{n-1} y(k-1) + \dots + a_0 y(k-n) = 0$
 - or substitute trial solution $y(k) = z^k$
into $a_n y(k+n) + \dots + a_1 y(k+1) + a_0 y(k) = 0$
note $y(k) = z^k; \quad y(k+1) = z^{k+1} = z y(k); \quad y(k-1) = z^{-1} y(k)$
 $a_n z^n y(k) + \dots + a_1 z y(k) + a_0 y(k) = 0$
 $(a_n z^n + \dots + a_1 z + a_0) y(k) = 0$
 $A(z) y(k) = 0$

3.3.3

so only solutions of characteristic equation $A(z)$ will work. Roots are real or conjugate pairs.

e.g. 0.8^k 1.2^k $(-3)^k$ $0.5^k \cos 2k$, $0.5^k \sin 2k$

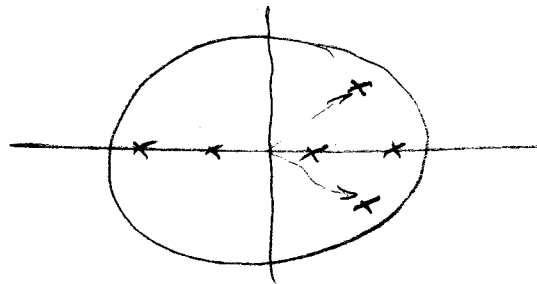
example $y(k+2) + 2y(k+1) + \frac{3}{4}y(k) = 0$

for homogeneous solution substitute $y(k) = z^k$

$$z^2 + 2z + \frac{3}{4} = 0 \quad z = -\frac{1}{2}, -\frac{3}{2}$$

$$y_h(k) = A\left(-\frac{1}{2}\right)^k + B\left(-\frac{3}{2}\right)^k$$

- roots must satisfy $|z| < 1$ for strictly stable system. complex roots or negative real roots give oscillatory solution



radius determines speed of decay

angle determines oscillation frequency

- radius determines speed of decay (or growth), angle determines oscillation frequency:

$$z = a \pm jb = R e^{\pm j\theta}$$

$$z^k = R^k e^{\pm jk\theta}$$

$$R^k e^{jk\theta} + R^k e^{-jk\theta} = R^k \cos k\theta$$

- repeated roots give solutions like $(1+2k) 0.8^k$
- characteristic of finite order systems is the pulse response as a sum of real or complex geometric sequences.

- We will not follow time domain solutions in detail. If you need it, read De Russo, Roy and Choe State Variables for Engineers or Poularikis and Sealy, Signals and Systems.

- But as an example, here's how to get the pulse response.

What is $h(i)$ for $y(i) = 2y(i-1) + x(i) + x(i-1)$?

It is $y(i)$ for $x(i) = \delta(i)$. Since $\delta(i) = 0$ for $i \geq 1$,

$y(i)$ is homogeneous solution for $i \geq 1$, since $\delta(i) = 0$ (a constant) for $i \geq 1$, which means the particular solution is const. at 0.

i	$y=h$	x
-1	0	0
0	1	1
1	3	0
2	6	0

homogeneous solution

$$y(i) - 2y(i-1) = 0$$

try β^i

$$\beta^i - 2\beta^{i-1} = \beta^{i-1}(\beta - 2) = 0 \Rightarrow \beta = 2$$

$$y_h = A 2^i$$

Comparison for $i \geq 1$ (where $\delta(i)$ becomes const) gives $A = 3/2$.

$$\text{Therefore } h(i) = \begin{cases} 1, & i=0 \\ \frac{3}{2} 2^i, & i \geq 1 \end{cases} \quad \leftarrow \begin{array}{l} \text{special value before} \\ \delta(i) \text{ becomes zero} \end{array}$$

$$\text{or } h(i) = \frac{3}{2} 2^i - \frac{1}{2} \delta(i)$$

example system described by

$$y(k) + \frac{1}{4} y(k-1) - \frac{3}{8} y(k-2) = x(k) - 2x(k-2)$$

with non zero IC's $y(-1) = 1$, $y(-2) = 2$.

What is response to step input? (not $g(k)$ since IC's $\neq 0$)

$y_p(k)$: $x(k)$ is a constant. Try $y_p(k) = C$, also constant.
so $y_p(k-2) = y_p(k-1) = y_p(k)$, $k \geq 2$

$$C(1 + \frac{1}{4} - \frac{3}{8}) = 1 - 2 \quad C = -8/7 = y_p(k)$$

$y_h(k)$: try $y_h(k) = z^k$ $y_h(k-1) = z^{-1} y_h(k)$

$$y_h(k) + \frac{1}{4} y_h(k-1) - \frac{3}{8} y_h(k-2) = 0$$

$$(1 + \frac{1}{4} z^{-1} - \frac{3}{8} z^{-2}) y_h(k) = 0$$

$$z^2 + \frac{1}{4} z - \frac{3}{8} = 0 \quad z = \frac{1}{2}, -\frac{3}{4}$$

$$y_h(k) = A\left(\frac{1}{2}\right)^k + B\left(-\frac{3}{4}\right)^k$$

total $y(k) = y_p(k) + y_h(k) = -\frac{8}{7} + A\left(\frac{1}{2}\right)^k + B\left(-\frac{3}{4}\right)^k$

now pick A, B to satisfy IC's:

k	y	x
-2	2	0
-1	1	0
0	1.5	1
1	1	1
2	-0.6875	1
3	-0.453125	1

$$y(0) = -\frac{1}{4} y(-1) + \frac{3}{8} y(-2) + x(0) - 2x(-2)$$

fit $y(0) = 1.5$ $y(1) = 1$

$$-\frac{8}{7} + A\left(\frac{1}{2}\right)^0 + B\left(-\frac{3}{4}\right)^0 = 1.5$$

$$-\frac{8}{7} + A\left(\frac{1}{2}\right)^1 + B\left(-\frac{3}{4}\right)^1 = 1$$

$$A + B = 2.64$$

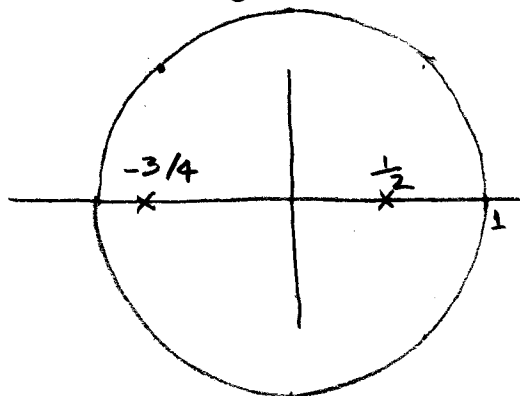
$$\frac{1}{2}A - \frac{3}{4}B = 2.14$$

solving $A + B = 2.64$
 $1.25A = 4.12$

$$A = 3.3 \quad B = -0.66$$

and $y(k) = -1.14 + 3.3\left(\frac{1}{2}\right)^k - 0.66\left(-\frac{3}{4}\right)^k \quad k \geq 0$

check $y(2) = -1.14 + 3.3\left(\frac{1}{2}\right)^2 - 0.66\left(-\frac{3}{4}\right)^2$
 $= -0.6875$



roots of characteristic equation are in the unit circle, hence stable.

Note: if we had wanted $g(k)$, unit step response, we would have set IC's to zero. In fact, let's get the response partitioned as zero state and zero input responses.

zic: $y(k) + \frac{1}{4}y(k-1) - \frac{3}{8}y(k-2) = 0 \quad k \geq 0$

$y(-1) = 1 \quad y(-2) = 2 \quad \text{ICs}$

This is a homogeneous solution $A\left(\frac{1}{2}\right)^k + B\left(-\frac{3}{4}\right)^k$
 so fit to ICs:

$$A\left(\frac{1}{2}\right)^{-1} + B\left(-\frac{3}{4}\right)^{-1} = 1$$

$$2A - \frac{4}{3}B = 1$$

$$A\left(\frac{1}{2}\right)^{-2} + B\left(-\frac{3}{4}\right)^{-2} = 2$$

$$4A + \frac{16}{9}B = 2$$

$$A = \frac{1}{2} \quad B = 0$$

$$y_{zi}(k) = \left(\frac{1}{2}\right)^{k+1}$$

$$zsr: y(k) + \frac{1}{4} y(k-1) - \frac{3}{8} y(k-2) = x(k) - 2x(k-2)$$

$$\text{with } y(-1) = 0 \quad y(-2) = 0$$

The solution is sum of particular and homogeneous solutions, with coefficients selected to fit zero ICs.

$$\text{From previous work } y(k) = -\frac{8}{7} + C\left(\frac{1}{2}\right)^k + D\left(-\frac{3}{4}\right)^k$$

k	y	x
-2	0	0
-1	0	0
0	1	1
1	$\frac{3}{4}$	1
2	$-\frac{13}{16}$	1
3	-0.516	1

$$y(0) = -\frac{1}{4} y(-1) + \frac{3}{8} y(-2) + x(0) - 2x(-2)$$

$$k=0 \quad C + D = \frac{8}{7} + 1 \quad C = \frac{14}{5} \quad D = -\frac{23}{35}$$

$$k=1 \quad \frac{1}{2}C - \frac{3}{4}D = \frac{8}{7} + \frac{3}{4} \quad = 2.8 \quad = -0.657$$

$$y_{zs}(k) = -\frac{8}{7} + \frac{14}{5}\left(\frac{1}{2}\right)^k - \frac{23}{35}\left(-\frac{3}{4}\right)^k, \quad k \geq 0$$

$$\begin{aligned} \text{Check } y_{zs}(k) + y_{zi}(k) &= -\frac{8}{7} + \frac{33}{10}\left(\frac{1}{2}\right)^k - \frac{23}{35}\left(-\frac{3}{4}\right)^k \\ &= -1.14 + 3.3\left(\frac{1}{2}\right)^k - 0.66\left(-\frac{3}{4}\right)^k \end{aligned}$$

as before!

Example This one shows how to deal with complex roots in the characteristic polynomial.

Solve

$$y(k) - 1.6y(k-1) + 0.89y(k-2) = 2x(k) - x(k-2), \quad k \geq 0$$

for "at rest" ICs $y(-2)=0$ $y(-1)=0$

if $x(k) = u(k)$, the unit step.

Particular solution

The input is a constant ($=1$), so try $y_p(k) = C$, another constant. Substitute into the ΔE :

$$C - 1.6C + 0.89C = 2 \cdot 1 - 1 \cdot 1$$

which gives $C = 3.45$. Therefore

$$y_p(k) = 3.45$$

Homogeneous solution

The homogeneous ΔE is

$$y(k) - 1.6y(k-1) + 0.89y(k-2) = 0$$

Trial solution $y(k) = \beta^k$. Substitute and find values of β that work.

$$\beta^k - 1.6\beta^{k-1} + 0.89\beta^{k-2} = 0$$

$$\text{or } \beta^{k-2}(\beta^2 - 1.6\beta + 0.89) = 0$$

We could choose $\beta^{k-2} = 0$, but that's equivalent to $\beta = 0$ or $y_h = 0$, not a "solution" in any useful sense.

$$\text{So } \beta^2 - 1.6\beta + 0.89 = 0$$

$$\text{and } \beta_1 = 0.8 + 0.5j \quad \beta_2 = \beta_1^* = 0.8 - 0.5j$$

Any linear combination of these solutions satisfies the homogeneous ΔE , so

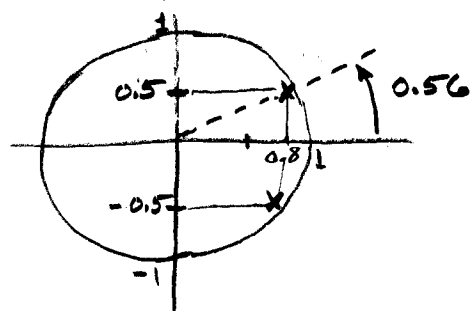
$$y_h(k) = D_1 \beta_1^k + D_2 \beta_2^k = D_1 \beta_1^k + D_2 (\beta_1^*)^k$$

This is more general than necessary: the input is real and the coefficients of the ΔE are real, hence $y_h(k)$ is real. The only way we can guarantee this for all k is $D_2 = D_1^*$

That makes $y_h(k) = D_1 \beta_1^k + D_1^* (\beta_1^*)^k = \text{Re}[D \beta_1^k]$, $D = 2D_1$

This is a good time to convert the root to polar notation

$$\beta_1 = 0.8 + 0.5j = 0.94 e^{j0.56}$$



Therefore

$$\begin{aligned} y_h(k) &= \text{Re}[D \beta_1^k] = \text{Re}[D 0.94^k e^{j0.56k}] \\ &= 0.94^k \text{Re}[D e^{j0.56k}] \end{aligned}$$

Although this version of $y_h(k)$ is useful for many purposes, it is often more comfortable to deal with a real expression that doesn't have hidden complex components.

So expand D in one of two ways:

polar $D = R e^{j\theta}$,

$$\begin{aligned} \text{so } y_h(k) &= 0.94^k \text{Re}[R e^{j\theta} e^{j0.56k}] = R 0.94^k \text{Re}[e^{j0.56k + \theta}] \\ &= R 0.94^k \cos(0.56k + \theta) \end{aligned}$$

rectangular $D = A + jB$

$$\begin{aligned} \text{so } y_h(k) &= 0.94^k \text{Re}[(A + jB)(\cos(0.56k) + j \sin(0.56k))] \\ &= 0.94^k (A \cos(0.56k) - B \sin(0.56k)) \end{aligned}$$

Total solution and fit to ICs

$$y(k) = y_p(k) + y_h(k) = 3.45 + 0.94^k (A \cos(0.56k) - B \sin(0.56k)) \quad k \geq 0$$

We'll fit A, B to $y(k)$ values at $k=0, 1$

so now we have to calculate those values. Rearranging the AE,

$$y(k) = 1.6 y(k-1) - 0.89 y(k-2) + 2x(k) - x(k-2).$$

k	x	y
-2	0	0
-1	0	0
0	1	2
1	1	5.2

x template of coefficients	
-1	-0.89
0	1.6
2	

$$k=0: 3.45 + 0.94^0 (A \cos(0.56 \cdot 0) - B \sin(0.56 \cdot 0)) = 2$$

$$3.45 + A = 2$$

$$A = -1.45$$

$$k=1: 3.45 + 0.94 (A \cos(0.56) - B \sin(0.56)) = 5.2$$

$$3.45 + 0.80A - 0.5B = 5.2$$

$$B = -5.82$$

Hence

$$y(k) = 3.45 + 0.94^k (-1.45 \cos(0.56k) + 5.82 \sin(0.56k)) \quad k \geq 0$$

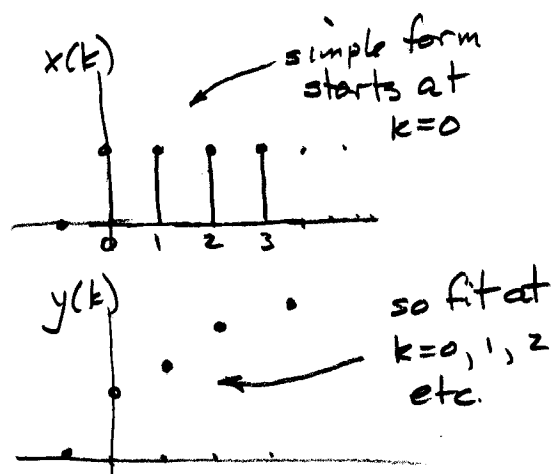
• Summary of ΔE solution methods

$$y(k) + a_{n-1}y(k-1) + \dots + a_0y(k-n) = b_mx(k+m-n) + \dots + b_0x(k-n)$$

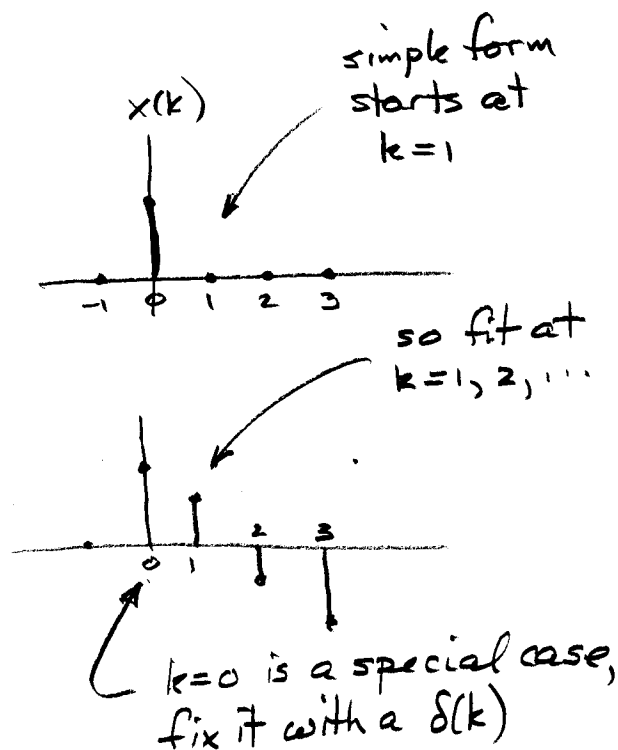
(a) Numerical: write $y(k) = \dots$ $k=0, 1, \dots$
and solve for $y(0), y(1), y(2), \dots$ successively

(b) Convolution: calculate $h(k)$ analytically (e.g. $h(k) = -(\frac{2}{3})^k$)
then $y(k) = \sum_{i=0}^k x(k-i)h(i)$

(c) Analytical: if $x(k)$ has a simple form (e.g. $\delta(k)$, $u(k)$, k , $2^{-k}\cos(0.1k + 0.5)$, etc, etc), then find $x_p(k)$.
Solution $y(k) = y_h(k) + y_p(k)$ and fit it where the simple $x(k)$ form begins.

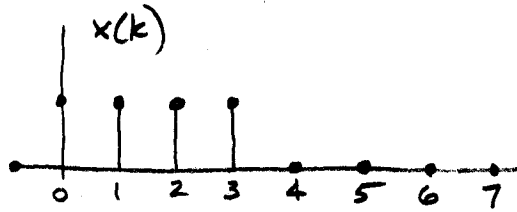


Iterate numerically to get $y(0), y(1), y(2)$ for fitting



(d) Mixed: if $x(k)$ does some wobbles before settling into a simple form.

e.g.



simple form starts at $k=4$
with constant (i.e. zero) input,
so $y_p(k) = C (=0)$ for $k \geq 4$

so (1) iterate numerically to get $y(0), \dots, y(3)$, then $y(4), y(5)$

(2) fit the analytical form (the params in $y_p(k) + y_h(k)$)
at $k=4, k=5, \dots, k=4+n-1$

This is valid for all $k \geq 4$.

(3) but it doesn't work for $k < 4$, so fix it up
with $\delta(k), \delta(k-1), \delta(k-2), \delta(k-3)$