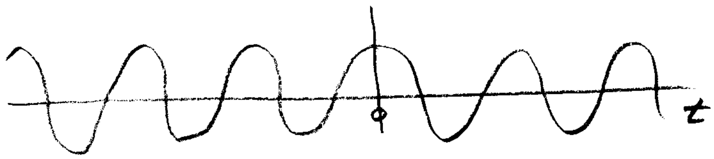
4.2 Frequency Response of Systems

- Frequency response is response of system to sinusoidal input after transients have died out (i.e., like the particular solution in finite order systems).
- Equivalently, it's the response to an "eternal sinusoid", one that has been running since $t = -\infty$



- Equivalently again, it's the response to the "eternal exponential" $x(t) = e^{j2\pi ft}$, $-\infty < t < \infty$.

So calculate it by convolution:

$$y(t) = \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha = \int_{-\infty}^{\infty} h(\alpha)\, e^{j2\pi f(t-\alpha)}\, d\alpha$$

$$= e^{j2\pi ft} \int_{-\infty}^{\infty} h(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha = e^{j2\pi ft} H(f)$$

This "self reproducing" property* of complex exponentials in convolution is the key to their value.

The function $H(f)$ gives the output phasor as a function of frequency, if the input phasor is 1. This function <u>is</u> the frequency response.

It is the complex "gain" of the system at frequency f.

*they are eigenfunctions of convolution

- Here's how to relate it to real signals.

Example: $x(t) = \cos(2\pi f t) = \mathrm{Re}\left[e^{j2\pi f t}\right], \; -\infty < t < \infty$

so $$y(t) = \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha = \int_{-\infty}^{\infty} h(\alpha)\, \mathrm{Re}\left[e^{j2\pi f(t-\alpha)}\right] d\alpha$$

$$= \mathrm{Re}\left[\int_{-\infty}^{\infty} h(\alpha)\, e^{j2\pi f(t-\alpha)}\, d\alpha\right] = \mathrm{Re}\left[e^{j2\pi f t}\int_{-\infty}^{\infty} h(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha\right]$$

$$= \mathrm{Re}\left[e^{j2\pi f t} H(f)\right]$$

- And here is an example to show the power of frequency domain representation. Suppose the input is a sum of sinusoids:

$$x(t) = \mathrm{Re}\left[X_1 e^{j2\pi f_1 t}\right] + \mathrm{Re}\left[X_2 e^{j2\pi f_2 t}\right] + \mathrm{Re}\left[X_3 e^{j2\pi f_3 t}\right]$$

By superposition:

$$y(t) = \mathrm{Re}\left[X_1 H(f_1) e^{j2\pi f_1 t}\right] + \mathrm{Re}\left[X_2 H(f_2) e^{j2\pi f_2 t}\right] + \mathrm{Re}\left[X_3 H(f_3) e^{j2\pi f_3 t}\right]$$

$$= \mathrm{Re}\left[Y_1 e^{j2\pi f_1 t}\right] + \mathrm{Re}\left[Y_2 e^{j2\pi f_2 t}\right] + \mathrm{Re}\left[Y_3 e^{j2\pi f_3 t}\right]$$

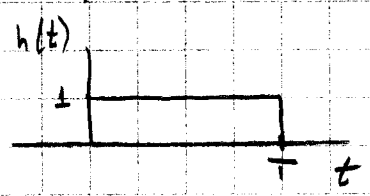
Summary — the output is sum of sinusoids at the same frequencies, with coefficients determined easily by

$$Y_k = X_k H(f_k)$$

scaled independently of each other

Note: $\omega = 2\pi f$, so frequency response is often written $H(\omega) = \int_{-\infty}^{\infty} h(\alpha)\, e^{-j\omega\alpha}\, d\alpha$

- Example — the moving window averager (MWA)



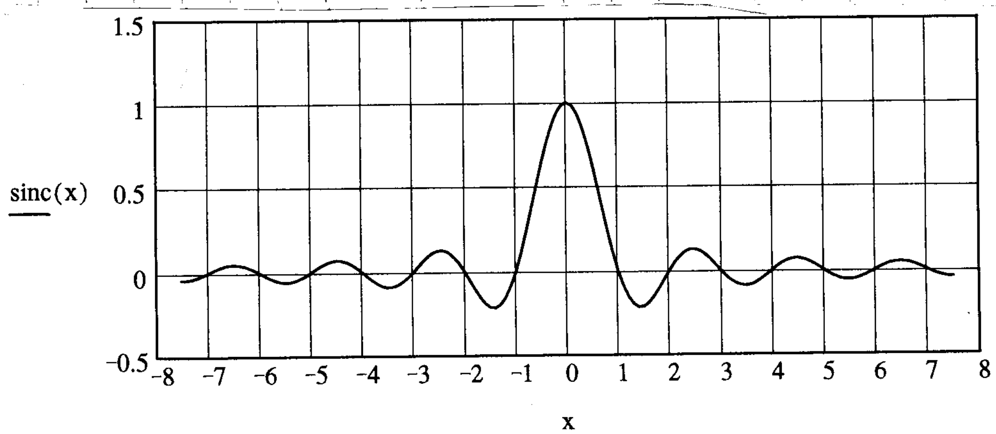
- first, the freq response (response to eternal exponential at f)

$$x(t) = e^{j2\pi ft}$$

$$y(t) = e^{j2\pi ft}\int_{-\infty}^{\infty} h(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha = e^{j2\pi ft}\int_0^T e^{-j2\pi f\alpha}\, d\alpha$$

$$= e^{j2\pi ft}\,\frac{1 - e^{-j2\pi fT}}{j2\pi f} = e^{j2\pi ft}\, e^{-j\pi fT}\,\frac{e^{j\pi fT} - e^{-j\pi fT}}{j2\pi f}$$

$$= e^{j2\pi ft}\, e^{-j\pi fT}\, T\,\mathrm{sinc}\, fT \qquad \mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$



- so response to eternal cosine $Re\left[e^{j2\pi ft}\right]$

is $Re\left[\underbrace{T\,\mathrm{sinc}(fT)\, e^{-j\pi fT}}_{H(f)}\, e^{j2\pi ft}\right]$

$$|H(f)| = T\,\mathrm{sinc}\, fT \qquad \measuredangle H(f) = -\pi fT = -\omega T/2$$

- Decaying exponentials are also self reproducing:

$$x(t) = e^{(\sigma+j\omega)t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha = \int_{-\infty}^{\infty} h(\alpha)\, e^{(\sigma+j\omega)t} e^{-(\sigma+j\omega)\alpha} d\alpha$$

$$= e^{(\sigma+j\omega)t} \int_{-\infty}^{\infty} h(\alpha)\, e^{-(\sigma+j\omega)\alpha} d\alpha$$

$$= e^{st} \int_{-\infty}^{\infty} h(\alpha)\, e^{-s\alpha} d\alpha$$

Note: I used ω, not f, for an easier link with Laplace

- The rest of Fourier analysis consists of representing time functions as sums (integrals) of exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{j2\pi kt/T}$$ Fourier series, periodic $x(t)$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df$$ Fourier transform, aperiodic $x(t)$

after which we can apply the frequency response to get the frequency makeup of the output

$$y(t) = \sum_{k=-\infty}^{\infty} X_k\, H(k/T)\, e^{j2\pi kt/T}$$ periodic

$$y(t) = \int_{-\infty}^{\infty} X(f)\, H(f)\, e^{j2\pi ft} df$$ aperiodic