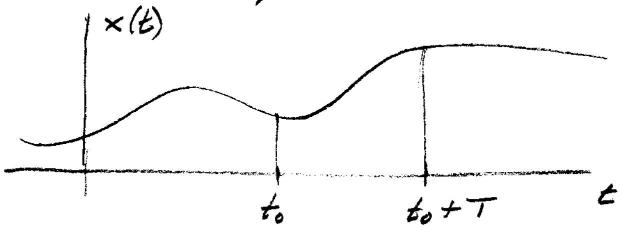
4.3 Fourier Series

H&vV 3.3

read "complex exponentials and orthogonality" on website

Over a finite interval, a function can be expressed as sum of sinusoids, each with its own amplitude and phase, at frequencies that are multiples of a fundamental.



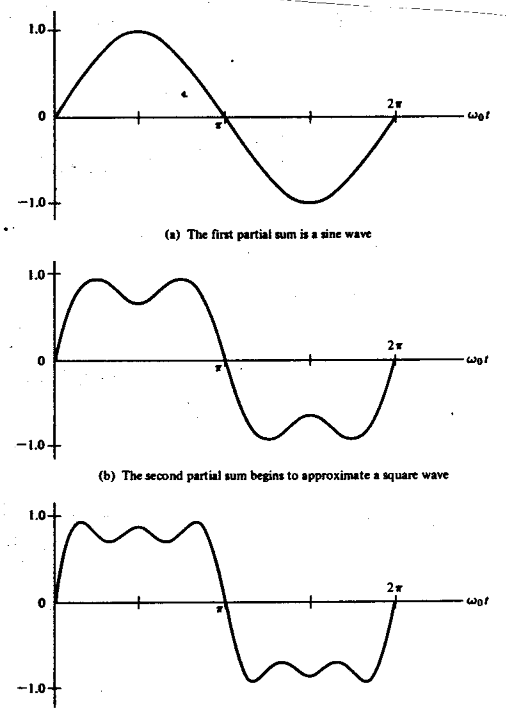
for $t_0 \le t \le t_0 + T$:

$$\hat{x}(t) = \sum_{n=0}^{\infty} R_n \cos\left(\frac{2\pi n t}{T} + \varphi_n\right)$$

$$= \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi n t}{T}\right) - B_n \sin\left(\frac{2\pi n t}{T}\right)$$

$$= \sum_{n=-\infty}^{\infty} X_n e^{j 2\pi n t / T} \qquad (X_{-n} = X_n^* \text{ for real } x(t))$$

$$R_n e^{j\varphi_n} = A_n + jB_n = X_n$$

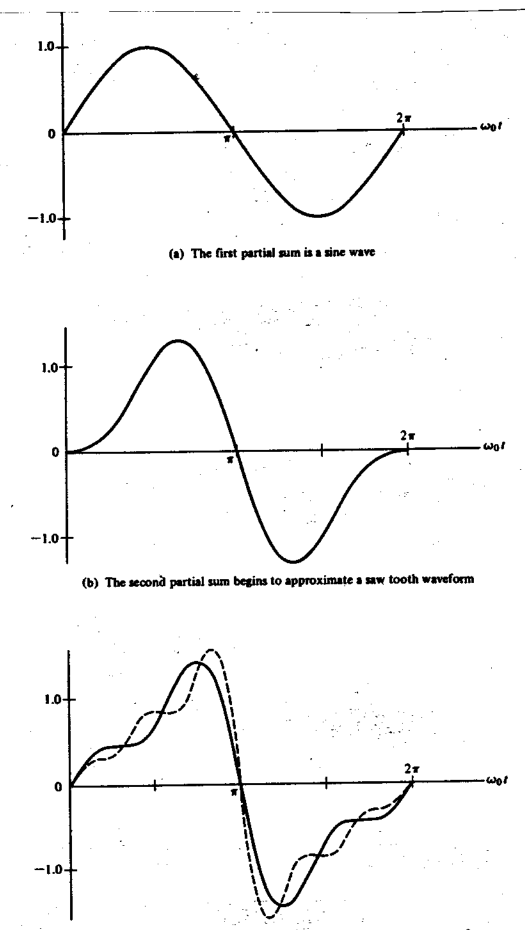


(a) The first partial sum is a sine wave

(b) The second partial sum begins to approximate a square wave

(c) The third partial sum provides a better approximation to a square wave

FIGURE 3-1. The series of Example 3-1 shows how a trigonometric series approximates a square wave (only one period shown).



(a) The first partial sum is a sine wave

(b) The second partial sum begins to approximate a saw tooth waveform

(c) The third and fifth partial sums better approximate a saw tooth waveform

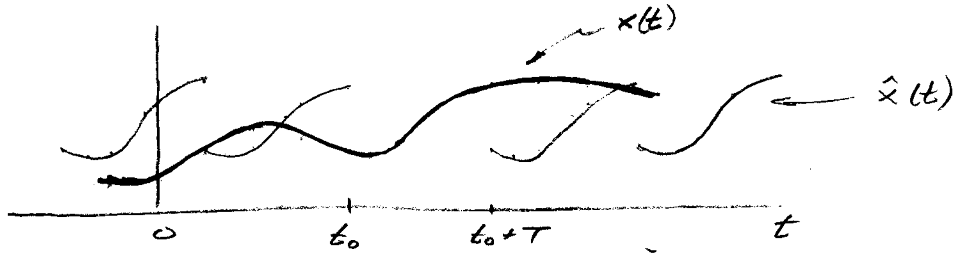
FIGURE 3-2. The trigonometric series of Example 3-2 illustrates approximation of a sawtooth waveform (only one period shown).

- If we select the coefficients properly then:

  the series $\hat{x}(t) = x(t)$ the original function, in $t_0 \le t \le t_0+T$.

Outside the representation interval they may differ, since

$$\hat{x}(t) = \sum_{n=0}^{\infty} R_n \cos\left(\frac{2\pi n t}{T} + \varphi_n\right)$$ is periodic in t, period T



(F series uses eternal exponentials)

If $x(t)$ is itself periodic with period T, then $\hat{x}(t) = x(t)$.

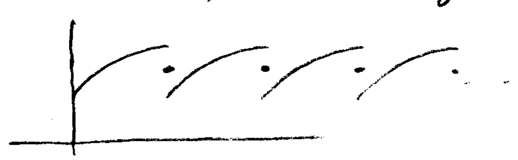
- Actually $\hat{x}(t) \to x(t)$ in this sense:

  if $\hat{x}_n(t) = \sum_{k=0}^{n} R_k \cos(k\omega_0 t + \varphi_k)$ $\qquad \left(\omega_0 = \frac{2\pi}{T},\ f_0 = \frac{1}{T}\right)$

  and the error $e_n(t) = x(t) - \hat{x}_n(t)$

  then $\lim_{n\to\infty} \int_{t_0}^{t_0+T} e_n^2(t)\,dt = 0$

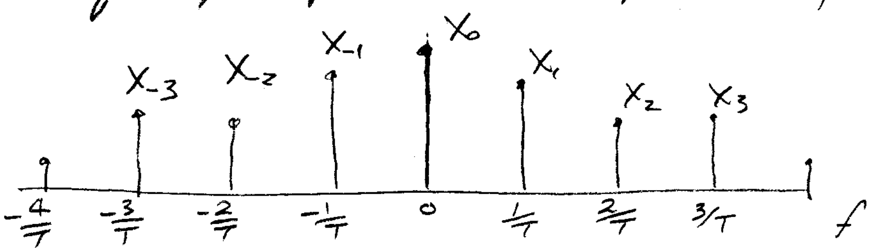
- At discontinuities, it converges to average of left and right limits:



Can have only a finite number of step discontinuities in the representation interval.

- The frequency components are spaced by $\frac{1}{T} = f_0$ Hz

  $\frac{2\pi}{T} = \omega_0$ rps



- Here's how to calculate the coefficients. Easiest in exponential form:

$$x(t) \text{ "=" } \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} = \hat{x}(t)$$

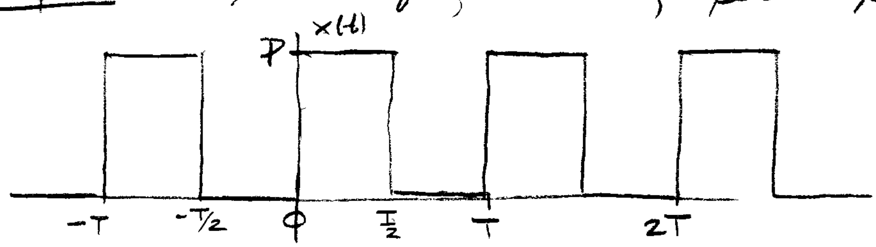
$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} X_k e^{j(k-n)\omega_0 t}$$

$$\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} X_k \underbrace{\int_{t_0}^{t_0+T} e^{j(k-n)\omega_0 t} dt}_{\begin{array}{ll} T & k=n \\ 0 & k \neq n \end{array}}$$

$$= T X_n$$

$$\left.\begin{array}{l} X_n = \dfrac{1}{T} \displaystyle\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t} dt \\ x(t) = \displaystyle\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \end{array}\right\} \text{Fourier series pair}$$

<u>example</u> one/zero square wave, peak to peak P



choose symmetric interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$ for convenience

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t} dt = \frac{P}{T} \int_0^{T/2} e^{-jn2\pi t/T} dt$$

$$= \frac{P/T}{-jn\,2\pi/T}\left(e^{-jn\pi} - 1\right)$$

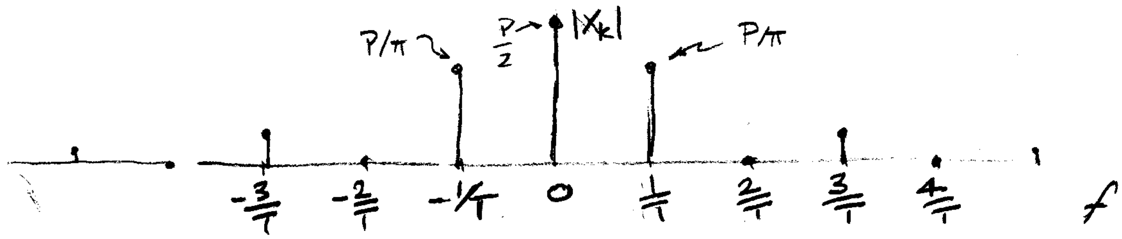
$$= \begin{cases} P/2 & n=0 \\ 0 & \text{other even } n \\ -\dfrac{jP}{\pi n} & n \text{ odd} \end{cases}$$

so $$x(t) = \frac{P}{2} + \sum_{\text{odd } n} \left(-\frac{jP}{\pi n}\right) e^{j2\pi nt/T}$$

$$= \frac{P}{2} + \sum_{\text{odd } n>0} 2\,\mathrm{Re}\left[\frac{-jP}{\pi n} e^{j2\pi nt/T}\right]$$

since the $-n$ term is the conjugate of the $+n$ term

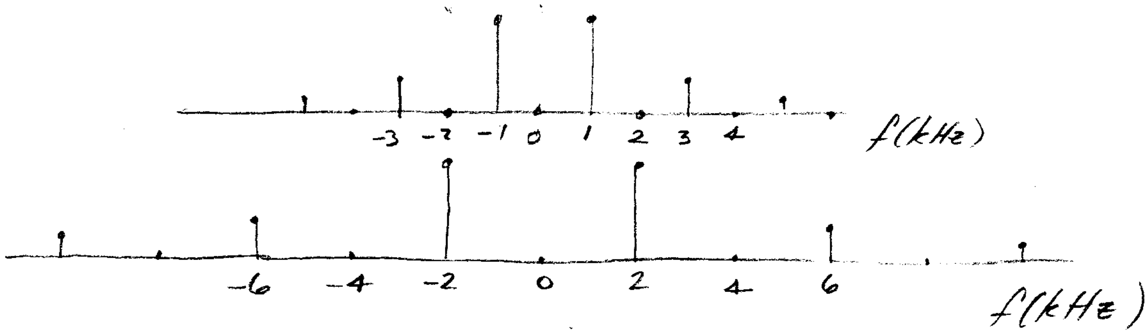
$$= \frac{P}{2} + \sum_{\text{odd } n>0} \frac{2P}{\pi n} \sin\left(2\pi n \frac{t}{T}\right)$$



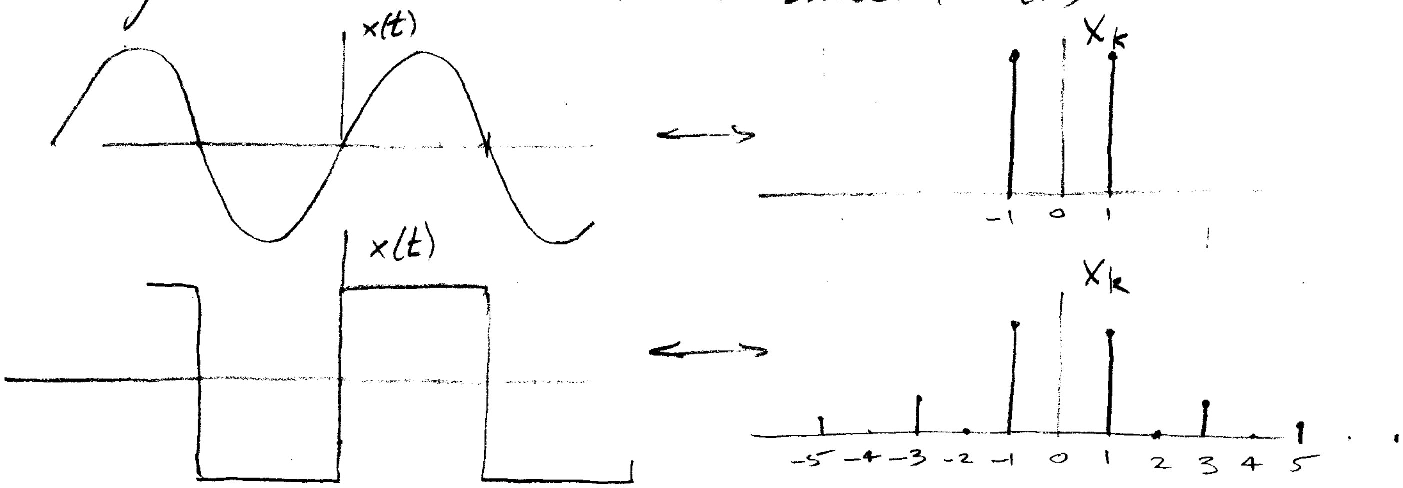
- The sketch shows the amplitude distribution in f
  - no even harmonics for symmetric square wave
  - odd harmonics are in proportion $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$
  - the component at $f=0$ is the dc, or average, value
- In terms of amplitude $A = \frac{1}{2}P$ (the pk-pk) we have

$$x(t) = A + \sum_{\text{odd } n>0} \frac{4A}{\pi n} \sin(2\pi n t/T)$$

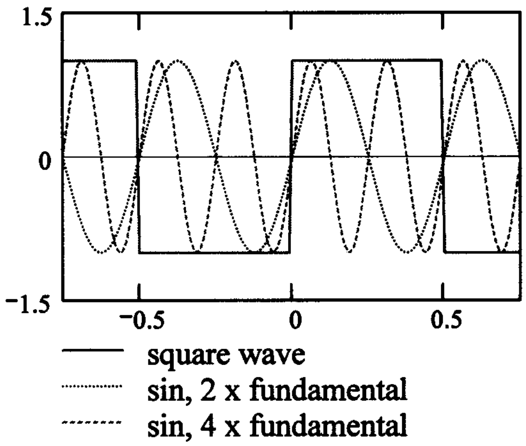
- Quickly varying $x(t)$ have greater bandwidth than slowly varying:
  - A 1kHz periodic signal has components at multiples of 1kHz. Speed it up to 2kHz and frequency components spread out twice as far

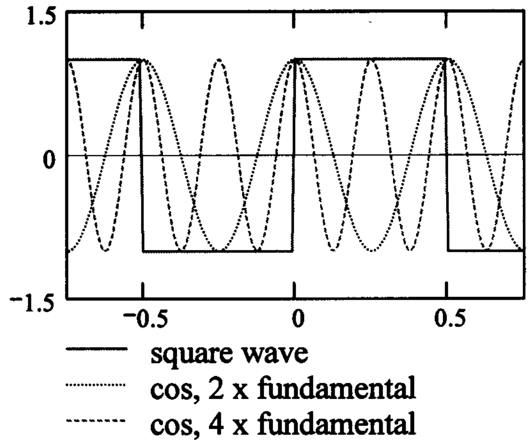


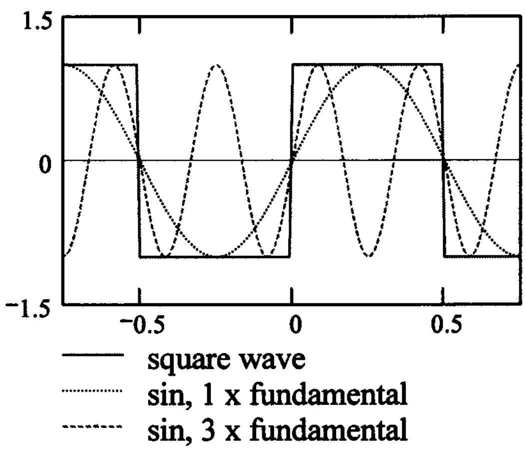
  - Even with the same period, jagged $x(t)$ have greater bandwidth than smooth $x(t)$

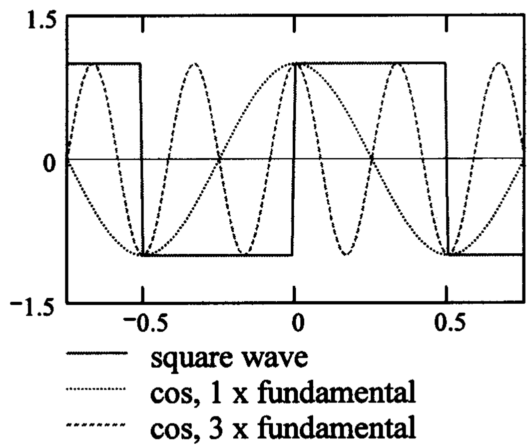


Only the odd coefficients of a square wave are non-zero:









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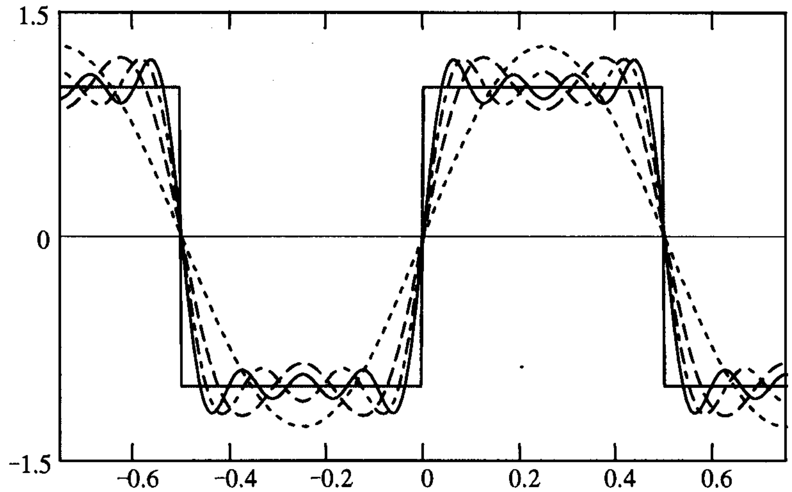
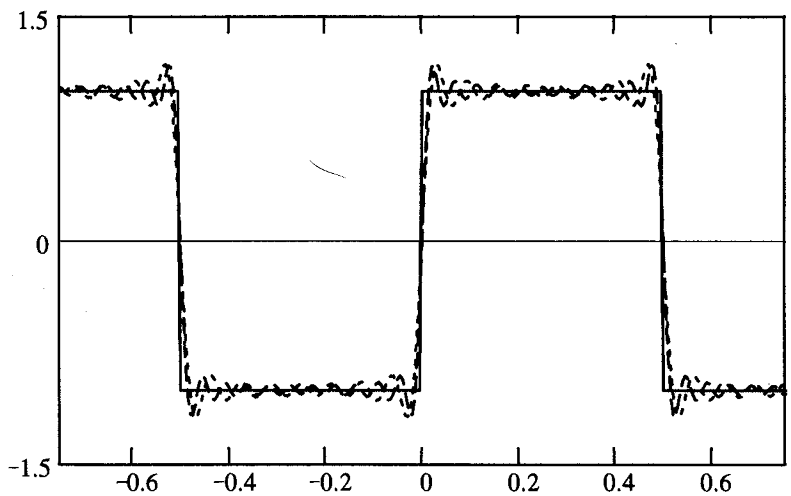
An illustration of the Gibbs phenomenon:

$$P := 1 \qquad x_{hat}(t,N) := \sum_{n=1}^{N} odd(n)\cdot\frac{4\cdot P}{\pi\cdot n}\cdot\sin\left(2\cdot\pi\cdot n\cdot\frac{t}{T}\right)$$



— square wave
- - - n=1 term
-- terms 1 and 3
-- terms 1,3 and 5
— terms 1,3,5 and 7



— square wave
~~n=1 term~~ 15
~~terms 1 and 3~~ 21