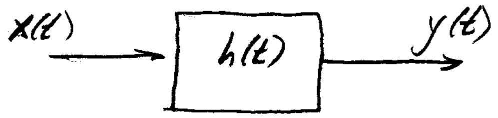
4.4 Applications and Properties of Fourier Series

read "Fourier series" from web site

- With F series, it's easy to calculate the response of a linear system to a periodic input $x(t)$



If $x(t) = \sum_k X_k e^{jk\omega_0 t}$ then we can calculate response to each term individually.

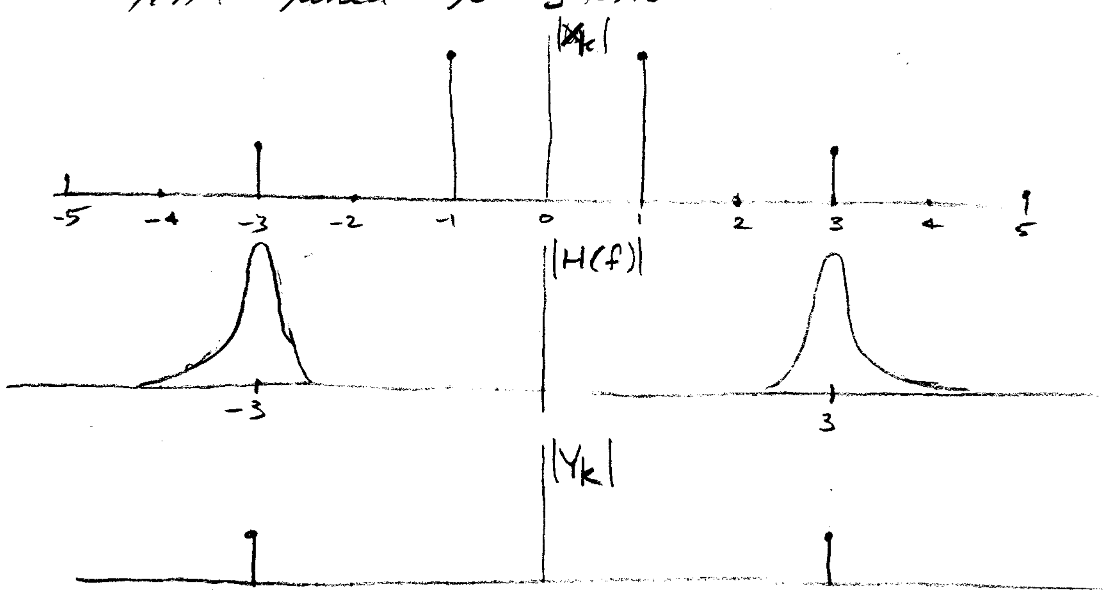
Response to eternal exponential $e^{jk\omega_0 t}$ is $H(k\omega_0) e^{jk\omega_0 t}$

$$y(t) = \sum_k X_k H(k\omega_0) e^{jk\omega_0 t}$$

$$= \sum_k Y_k e^{jk\omega_0 t} \qquad \left(Y_k = H(k\omega_0) X_k \right)$$

also periodic, with same period.

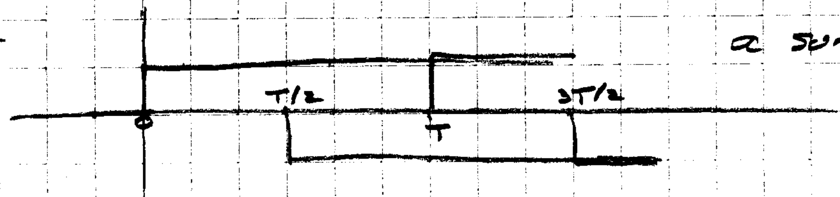
<u>example</u> 1 kHz square wave through narrowband filter tuned to 3 kHz



Tuned mechanical or electrical systems will resonate in response to harmonics of driving signal.

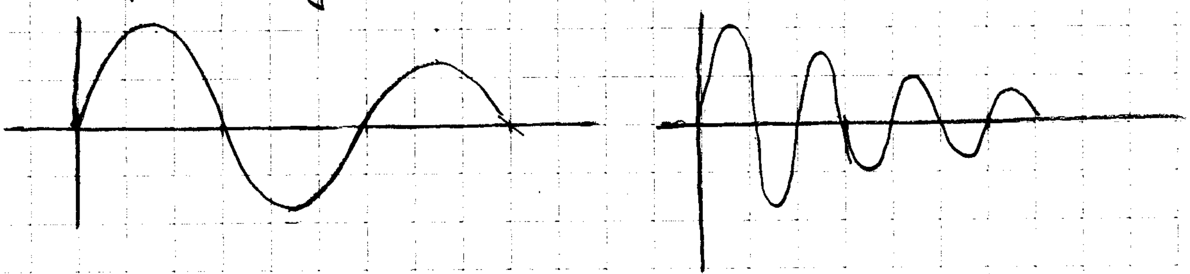
- Time domain explanation of resonance to harmonics:

square wave =

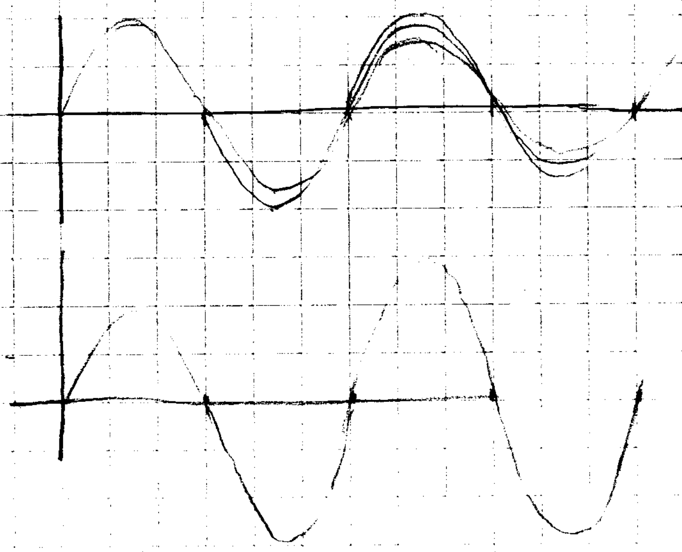


a sum of alternating steps

step responses of BPFs tuned to 1 and 3 × fundamental:

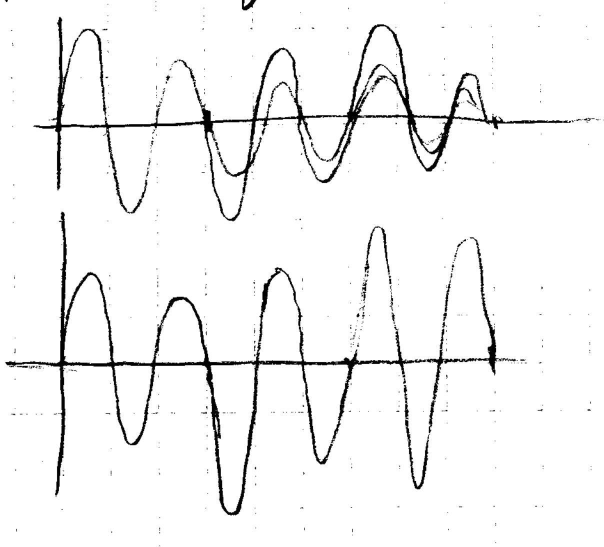


response to square wave:

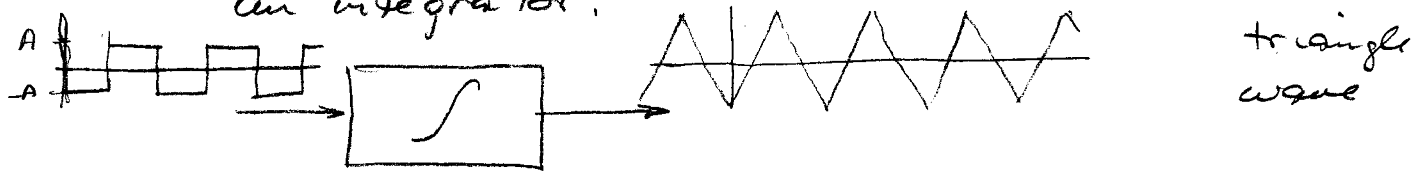


they reinforce

response to square wave:



<u>example</u> square wave, with no dc component, through an integrator.



The frequency response of an integrator is $\frac{1}{j\omega}$, so

$$x(t) = \sum_{\text{odd } n} \frac{-j2A}{\pi n} e^{j2\pi n t/T} \qquad \omega_0 = \frac{2\pi}{T}$$

$$y(t) = \sum_{\text{odd } n} \frac{-j2A}{\pi n \, j n\omega_0} e^{j2\pi n t/T}$$

$$= \sum_{\text{odd } n} -\frac{AT}{\pi^2 n^2} e^{j2\pi n t/T} = \sum_{\text{odd } n>0} -\frac{2AT}{\pi^2 n^2} \cos(2\pi n t/T)$$

- The energy in one cycle of $x(t)$ is obtained directly from the coefficients:

$$E = \int_0^T x^2(t)\,dt = \int_0^T \left(\sum_{k=-\infty}^{\infty} X_k e^{j2\pi k t/T} \right)^2 dt$$

$$= \int_0^T \sum_m \sum_n X_m X_n e^{j2\pi(m+n)t/T} dt$$

$$= T \sum_m X_m X_{-m}$$

$$= T \sum_{m=-\infty}^{\infty} |X_m|^2 = \int_0^T x^2(t)\,dt \qquad \text{Parseval's identity}$$

and average power

$$P = E/T = \sum_{m=-\infty}^{\infty} |X_m|^2$$

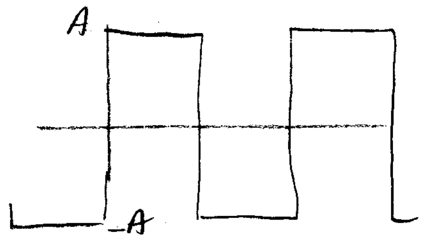
the sum of squared amplitude (power) at each frequency component.

<u>example</u> average power in dc free square wave:

$$X_m = -\frac{j2A}{\pi m}$$

so $$P = \sum_{\text{odd } m} \left| \frac{-j2A}{\pi m} \right|^2 = \sum_{\text{odd } m} \frac{4A^2}{\pi^2 m^2} = \frac{4A^2}{\pi^2} \sum_{\text{odd } m} \frac{1}{m^2}$$

$$= \frac{8A^2}{\pi^2} \underbrace{\sum_{\substack{\text{odd} \\ m>0}} \frac{1}{m^2}}_{\pi^2/8} = \frac{8A^2}{\pi^2} \frac{\pi^2}{8} = A^2$$



- If $x(t)$, $y(t)$ both periodic with same period, then
$$z(t) = x(t) + y(t)$$
has F series coefficients $Z_k = X_k + Y_k$.

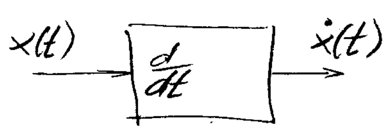
- If $x(t)$ has coefficients X_k, then coefficients of $\dot{x}(t)$ obtained by:

$$\dot{x}(t) = \frac{d}{dt} \sum_k X_k e^{j2\pi kt/T}$$

$$= \sum_k j\frac{2\pi k}{T} X_k e^{j2\pi kt/T}$$

the coefficients of $\dot{x}(t)$ are $j\frac{2\pi k}{T} X_k$

Checks with the idea of filtering. Frequency response of differentiator is $j\omega$:



coeff's at output are $H(k\omega_0)X_k$

$$= jk\frac{2\pi}{T}X_k$$

- An important result is the Fourier series for a product

$$z(t) = x(t)\, y(t)$$

if $x(t)$ and $y(t)$ have same period.

$$Z_n = \frac{1}{T}\int_0^T z(t)\, e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_0^T x(t)y(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T \Big(\sum_m X_m e^{jm\omega_0 t}\Big)\Big(\sum_k Y_k e^{jk\omega_0 t}\Big) e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T \sum_m \sum_k X_m Y_k \; e^{j\omega_0 t(m+k-n)}\,dt$$

$$= \frac{1}{T}\sum_m \sum_k X_m Y_k \underbrace{\int_0^T e^{j\omega_0 t(m+k-n)}\,dt}_{=\begin{cases} T, & m+k-n=0 \\ 0; & \text{otherwise}\end{cases}}$$

so only non zero terms have $m = n-k$

$$Z_n = \sum_k X_{n-k} Y_k \qquad \text{convolution !}$$

- Works the other way, too. If $z(t) = \int_0^T x(t-\alpha)\,y(\alpha)\,d\alpha$ then

$$Z_k = \frac{1}{T}\int_0^T z(t)\, e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_0^T\!\!\int_0^T x(t-\alpha)\,y(\alpha)\, e^{-jk\omega_0 t}\,dt\,d\alpha$$

$$= \frac{1}{T}\int_0^T\!\!\int_{0-\alpha}^{T-\alpha} x(\beta)\,y(\alpha)\, e^{-jk\omega_0(\beta+\alpha)}\,d\beta\,d\alpha \qquad (\beta = t-\alpha,\ t=\beta+\alpha,\ dt = d\beta$$

$$= \frac{1}{T}\int_0^T x(\beta)\, e^{-jk\omega\beta}\,d\beta \int_0^T y(\alpha)\, e^{-jk\omega_0\alpha}\,d\alpha$$

$$= T\, X_k Y_k$$

- multiply in $\begin{pmatrix}\text{time}\\ \text{frequency}\end{pmatrix}$ means convolve in $\begin{Bmatrix}\text{frequency}\\ \text{time}\end{Bmatrix}$

- Another important property:

  $x(t)$ is real iff $X_{-k} = X_k^*, \forall k$ "conjugate symmetry"

  - "if" Assume $X_{-k} = X_k^*$ for all k. Then

$$x(t) = \sum_k X_k e^{j2\pi kt/T} = X_0 + \sum_{k\geq 1}\left(X_k e^{j2\pi kt/T} + X_{-k} e^{-j2\pi kt/T}\right)$$

$$= X_0 + \sum_{k\geq 1}\left(X_k e^{j2\pi kt/T} + X_k^* e^{-j2\pi kt/T}\right)$$

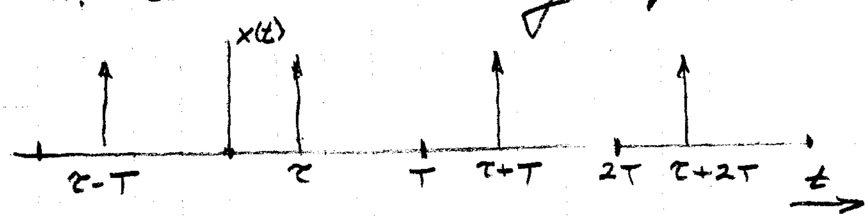
$$= X_0 + 2\sum_{k\geq 1} \mathrm{Re}\left[X_k e^{j2\pi kt/T}\right]$$

  - "only if" $X_k = \frac{1}{T}\int_0^T x(t) e^{-j2\pi kt/T} dt$

    so $X_k^* = \frac{1}{T}\int_0^T x(t) e^{j2\pi kt/T} dt = X_{-k}$

Put another way, the amplitude is an even function of k, phase is odd

- Ex. What is F series of impulse train?



$$x(t) = \sum_i \delta(t - iT - \tau)$$

$$x(t) = \sum_k X_k e^{j2\pi kt/T}$$

$$X_k = \frac{1}{T}\int_0^T \delta(t-\tau) e^{-j2\pi kt/T} dt = \frac{1}{T} e^{-j2\pi k\tau/T}$$

4.5 Summary of Fourier Series

- Any signal can be represented over a finite interval by a sum of sinusoids at multiples of the reciprocal interval size.

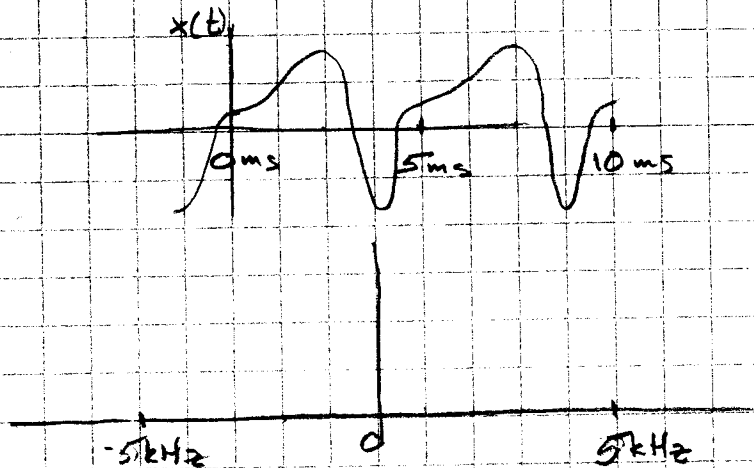
- If the signal is periodic, then:

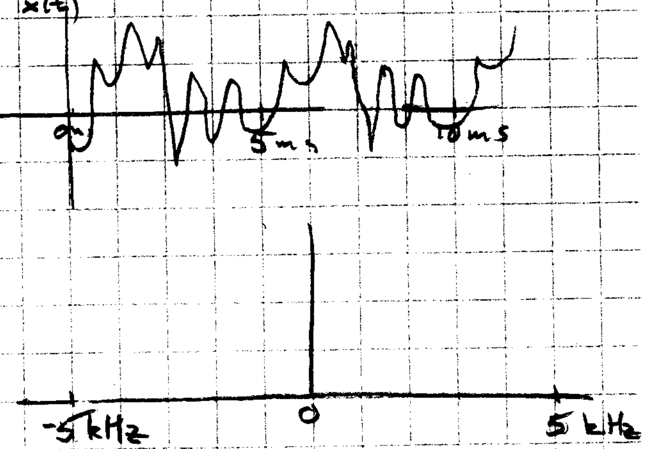
  - The representation is good everywhere

  - The signal can be considered as a superposition of equispaced tones.

  - Tone spacing and signal period scale inversely

- Number of significant components increases with finer detail in the signal.





- To calculate coeffs, end pts don't matter as long as right width

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-j2\pi nt/T} = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-j2\pi nt/T}$$

– The F series coefficients satisfy these properties:

- X_0 is the average, or dc, value
- $X_{-k} = X_k^*$ iff $x(t)$ is real — conj symm
- $\frac{1}{T}\int_0^T x^2(t)\,dt = \sum_k |X_k|^2$ — Parseval
- if $z(t) = x(t)y(t)$ then $Z_n = \sum_k X_{n-k} Y_k$ — mult-conv
- if $z(t) = x(t) \circledast y(t)$ then $Z_n = T X_n Y_n$ — conv-mult
- $\frac{1}{T}\int_0^T x(t)y(t)\,dt = \sum_k X_k Y_k^* = \sum_i X_i^* Y_i$ — inner product

  $\underline{\underline{\text{Pf}}}$ use mult-conv and conjugate symmetry

– If periodic $x(t)$ is convolved with some $h(t)$

$$y(t) = x(t) \circledast h(t)$$

then F series for $y(t) = \sum_k X_k H(2\pi k/T)\, e^{j2\pi kt/T}$

where $H(f)$ is the frequency response $H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi ft}\,dt$

– F series "vectorizes" periodic $x(t)$, to a discrete set of coeffs $\{X_k\}$ which we can consider as vector $\underline{X}$. This basis works well with convolution.

– F series is a "complete" basis. Define partial sum

$$\hat{x}_N(t) = \sum_{k=-N}^{N} X_k\, e^{j2\pi kt/T}$$

and error

$$e_N(t) = x(t) - \hat{x}_N(t)$$

Then $\displaystyle\lim_{N\to\infty} \int_0^T e_N^2(t)\,dt = 0$