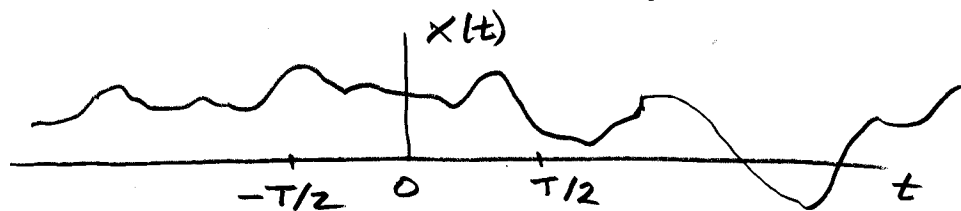


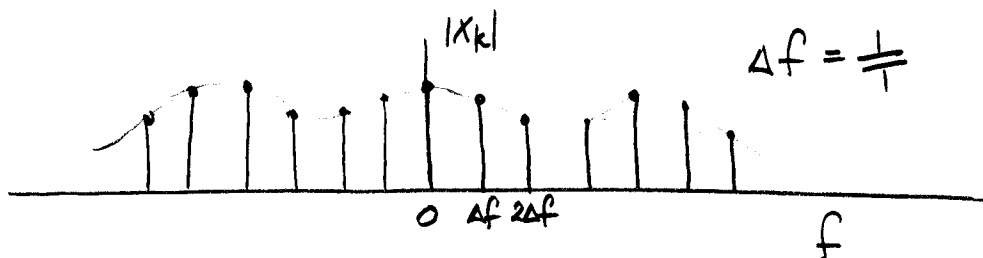
4.6 The Fourier Transform

4.6.1

- This section introduces the Fourier transform and its fundamental properties
- The F series shows that a periodic time function can be represented as a discrete function of frequency. The F transform extends this representation to non-periodic functions



- Fourier components are spaced by the reciprocal of the representation interval width.



As we represent more of $x(t)$ (i.e., T increases), the frequency components become more closely spaced

- Transition from series sum to an integral:

- the k^{th} frequency is $k\Delta f = k/T$

- denote $X(f_k) = TX_k$, so

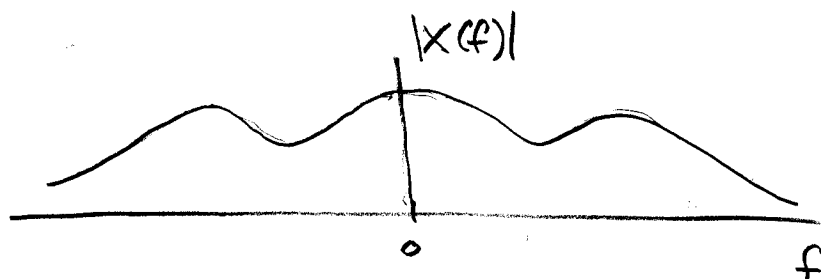
$$\text{analysis: } X(f_k) = TX_k = \int_{-T/2}^{T/2} x(t) e^{-j2\pi f_k t} dt$$

$$\text{synthesis: } x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k t/T} = \frac{1}{T} \sum_k X(f_k) e^{j2\pi k \Delta f t}$$

$$= \sum_k X(f_k) e^{j2\pi f_k t} \Delta f$$

$$\xrightarrow{T \rightarrow 0} \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

- As T increases, the bandwidth of $X(f)$ (the range of frequencies with significant power) changes very little - assuming $x(t)$ doesn't change its character (smoothness) as more of it is revealed by the widening window. Only the frequency resolution increases.



- We now have the Fourier transform pair (note duality)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

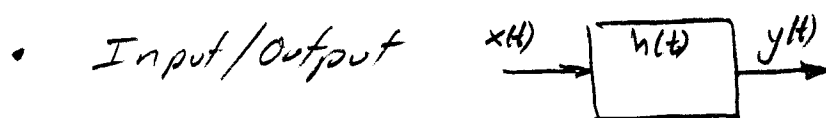
or

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

The forward xfm calculates the coefficient of $x(t)$ on the basis waveform $e^{j2\pi ft}$ and the inverse xfm represent $x(t)$ as a LC (weighted integral) of basis functions.



From the self-reproducing properties of complex exponentials, we might expect the coefficient of $y(t)$ wrt $e^{j2\pi ft}$ to be

$$Y(f) = H(f) X(f) \quad \text{as in F series}$$

Check it out:

$$\begin{aligned}
 y(t) &= \int x(\alpha) h(t-\alpha) d\alpha = \iint X(f) h(t-\alpha) e^{j2\pi \alpha f} d\alpha df \\
 &= \int X(f) \left(\int h(t-\alpha) e^{j2\pi \alpha f} d\alpha \right) df \\
 &= \int X(f) \int h(\beta) e^{j2\pi (t-\beta)f} d\beta df \\
 &= \int X(f) e^{j2\pi tf} \int h(\beta) e^{-j2\pi \beta f} d\beta df \\
 &= \int X(f) H(f) e^{j2\pi ft} df
 \end{aligned}$$

$$\text{so } Y(f) = X(f) H(f)$$

Note - what we have been calling "frequency response" is

$$H(f) = \mathcal{F}[h(t)]$$

- we have just showed that convolution in time corresponds to multiplication in freq.

How about mult in time? The F trans pair is symmetric, so if

$$z(t) = x(t) y(t)$$

then

$$Z(f) = \int x(f-\phi) Y(\phi) d\phi = \int x(\phi) Y(f-\phi) d\phi$$

but note in ω :

$$Z(\omega) = \frac{1}{2\pi} \int X(\omega-\eta) Y(\eta) d\eta = \frac{1}{2\pi} \int X(\eta) Y(\omega-\eta) d\eta$$

• Conclusions?

Mult in $\left\{ \begin{matrix} \text{time} \\ \text{freq} \end{matrix} \right\} \longleftrightarrow \text{conv in } \left\{ \begin{matrix} \text{freq} \\ \text{time} \end{matrix} \right\}$

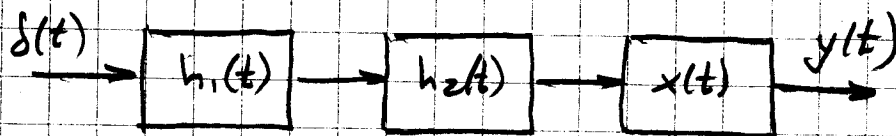
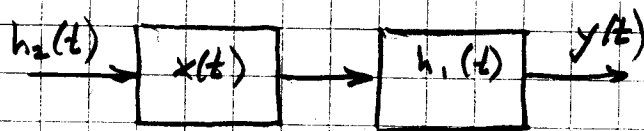
impulse response $h(t) \xrightarrow{\mathcal{F}} \text{frequency response } H(f)$
 $\xleftarrow{\mathcal{F}^{-1}}$

• Commutative property



$$Y(f) = X(f) H_1(f) H_2(f)$$

in any permutation



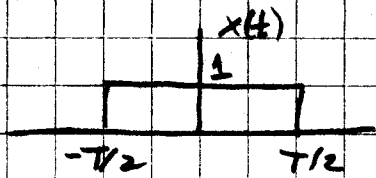
We saw this earlier in the time domain when we discussed convolution

• These are the fundamental properties of Fourier analysis, the reasons it is so attractive as a means of representing signals and systems.

• Building-Block Transforms

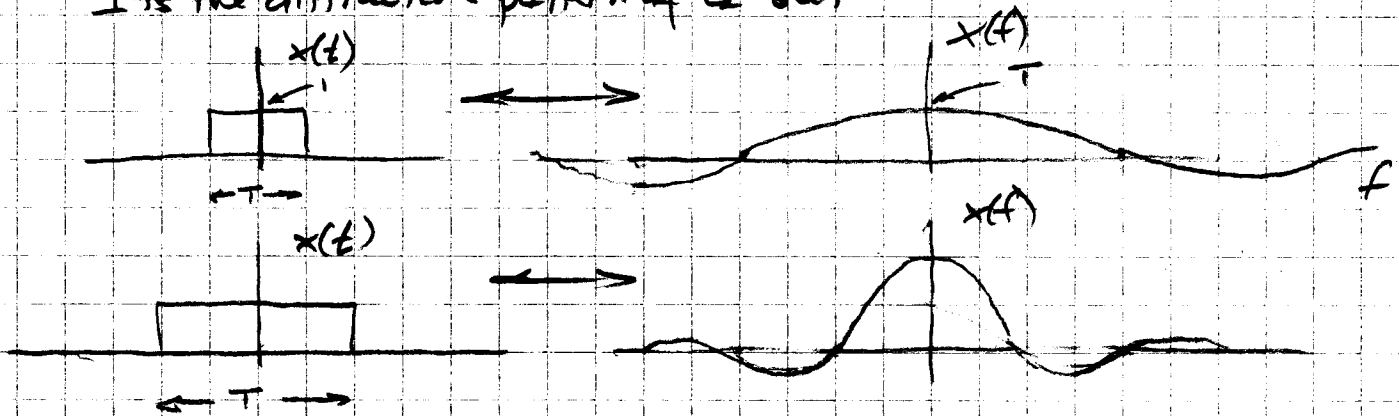
4.6.5

- Ex Fourier transform of a symmetric rect pulse

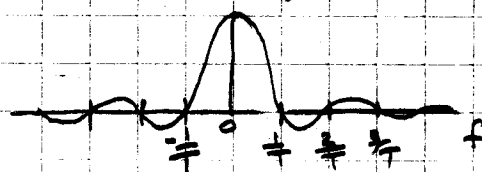
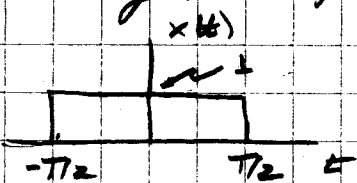


$$\begin{aligned}
 X(f) &= \int_{-\infty}^{\infty} e^{-j2\pi ft} dt = \int_{-T/2}^{T/2} e^{-j2\pi ft} dt \\
 &= \frac{1}{-j2\pi f} \left(e^{-j2\pi f T/2} - e^{j2\pi f T/2} \right) \\
 &= \frac{e^{-j\pi f T} - e^{j\pi f T}}{-j2\pi f} = -\frac{2j \sin(\pi f T)}{-2j\pi f} = T \operatorname{sinc}(fT)
 \end{aligned}$$

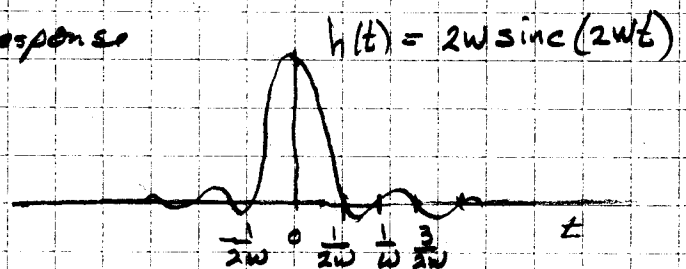
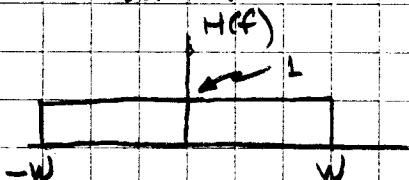
It's the diffraction pattern of a slit



- Use of time-frequency duality. We know that $x(f) = T \operatorname{sinc} fT$

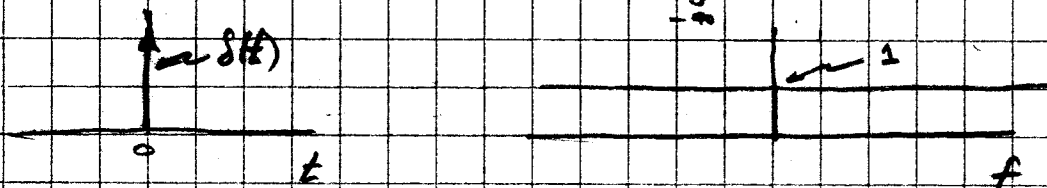


so ideal LPF has impulse response



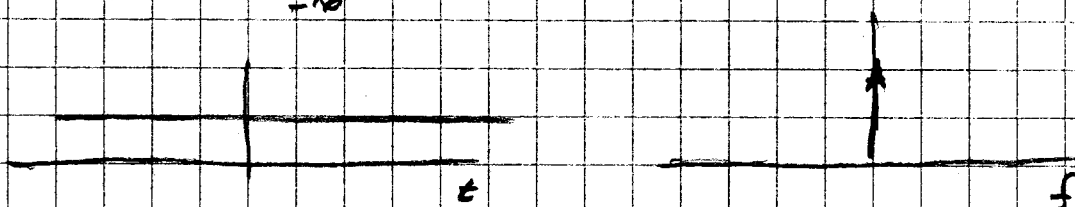
- Transform of an impulse:

$$x(t) = \delta(t) \longleftrightarrow X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = 1$$

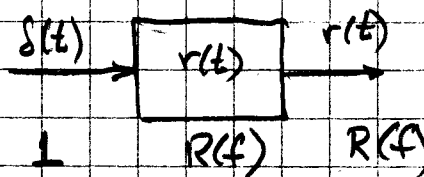


and

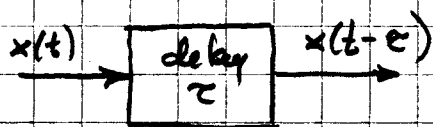
$$y(t) = \int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} df = 1 \longleftrightarrow Y(f) = \delta(f)$$



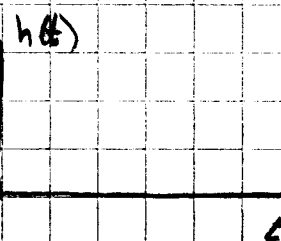
e.g. $r(t)$ is the same as



- What is frequency response of a delayer?

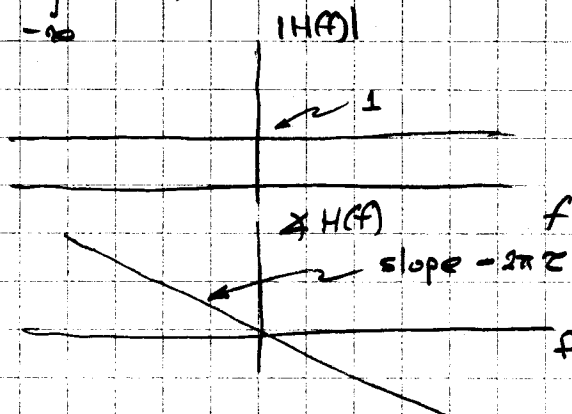


start with impulse resp



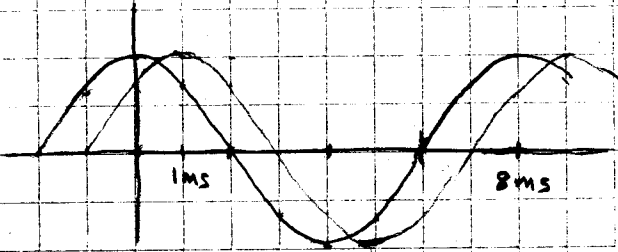
$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \delta(t - \tau) e^{-j2\pi ft} dt$$

$$= e^{-j2\pi f\tau}$$



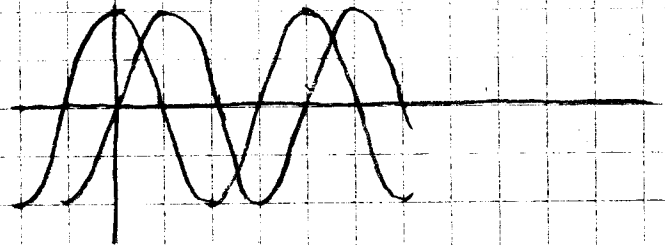
- Does that frequency response make sense? Sanity check:

For 125 Hz tone



delay of 1 ms is $45^\circ (\pi/4)$

For 250 Hz tone



delay of 1 ms is $90^\circ (\pi/2)$

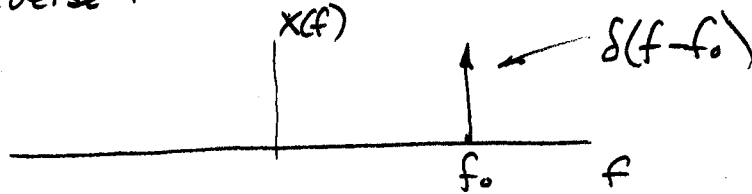
• Transform of a complex exponential $x(t) = e^{j2\pi f_0 t}$

- the hard way
$$X(f) = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} dt$$

$$= ??$$

- the easy way

inverse transform an offset impulse in frequency



$$\mathcal{F}^{-1}[\delta(f-f_0)] = \int_{-\infty}^{\infty} \delta(f-f_0) e^{j2\pi f t} df = e^{j2\pi f_0 t}$$

so
$$e^{j2\pi f_0 t} \longleftrightarrow \delta(f-f_0)$$

• A few properties of F transform

- area of $x(t)$ is $X(0)$; area of $X(f)$ is $x(0)$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \rightarrow X(0) = \int_{-\infty}^{\infty} x(t) dt$$

eg  $\rightarrow T \text{sinc}(fT)$ (also, area under $T \text{sinc}(fT)$ is 1)

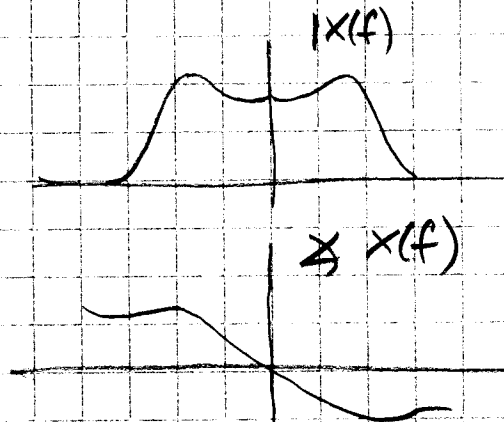
- domain reversal $x(-t) \leftrightarrow X(-f)$

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\ x(-t) &= \int_{-\infty}^{\infty} X(f) e^{-j2\pi ft} df \\ &= \int_{-\infty}^{\infty} X(-u) e^{j2\pi ut} (du) \\ &= \int_{-\infty}^{\infty} X(-u) e^{j2\pi ut} du \end{aligned}$$

- conjugate symmetry: $x(t)$ real iff $X^*(-f) = X(f)$

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\ x^*(t) &= \int_{-\infty}^{\infty} X^*(f) e^{-j2\pi ft} df \\ &= \int_{-\infty}^{\infty} X^*(-u) e^{j2\pi ut} du = x(t) \text{ since real} \end{aligned}$$

$$\text{so } X^*(-f) = X(f)$$



- energy and Parseval's theorem

we interpret $|x(t)|^2$ as power, hence energy as

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t) dt \\ &= \iint X(f) e^{j2\pi ft} x^*(t) dt df = \int X(f) \left(\int x^*(t) e^{j2\pi ft} dt \right) df \\ &= \int_{-\infty}^{\infty} |X(f)|^2 df \end{aligned}$$

so we compute energy the same way in both domains

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \quad \text{Parseval}$$

and we interpret $|X(f)|^2$ as energy spectral density

- reciprocal scaling. suppose $x(t) \longleftrightarrow X(f)$

$$\begin{aligned} \text{then } \mathcal{F}[x(at)] &= \int x(at) e^{-j2\pi ft} dt = \frac{1}{a} \int x(u) e^{-j2\pi \frac{f}{a} u} du \\ &= \frac{1}{a} X\left(\frac{f}{a}\right) \quad \text{where } a > 0 \end{aligned}$$

speedup (compress) if $a > 1$
slow down (dilate) if $a < 1$

- F transform and differentiation.

4.6.10

What is the F transform of $\dot{x}(t)$ ($\frac{dx}{dt}$) ?

If $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$

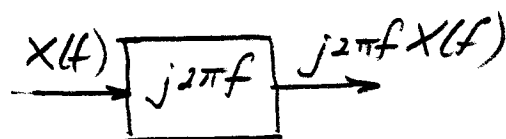
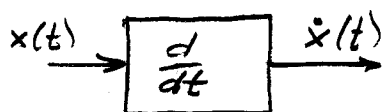
then $\dot{x}(t) = \int_{-\infty}^{\infty} X(f) j2\pi f e^{j2\pi ft} df$

that is

$$\dot{x}(t) \leftrightarrow j2\pi f X(f) \quad (\text{or } j\omega X(\omega))$$

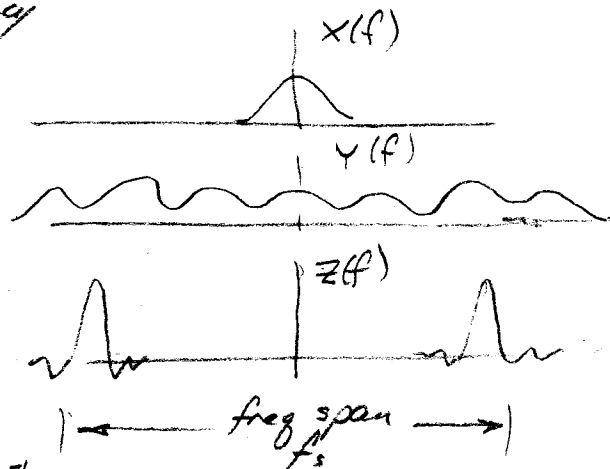
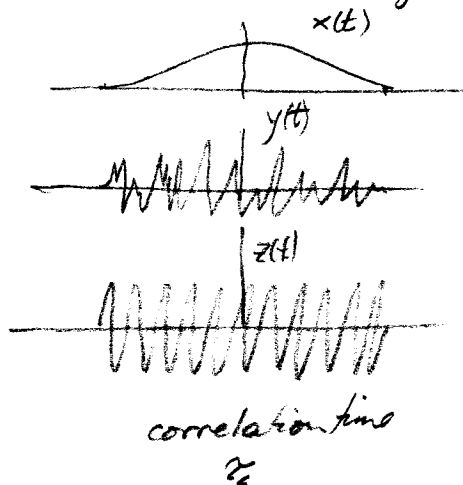
emphasizes high frequencies

so a differentiator:



(freq resp of differentiator)

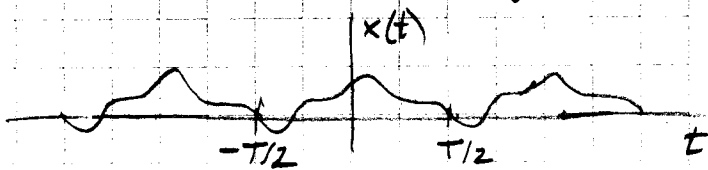
- apart from the precise inverse relation in time-frequency scaling, there is a useful approximate relationship:
the fine structure of a time signal (not the duration alone) determines the highest frequency



$$\tau_c \sim f_s^{-1}$$

- and vice versa: $f_c \sim t_s^{-1}$ (f_c is bandwidth for band-pass signal)

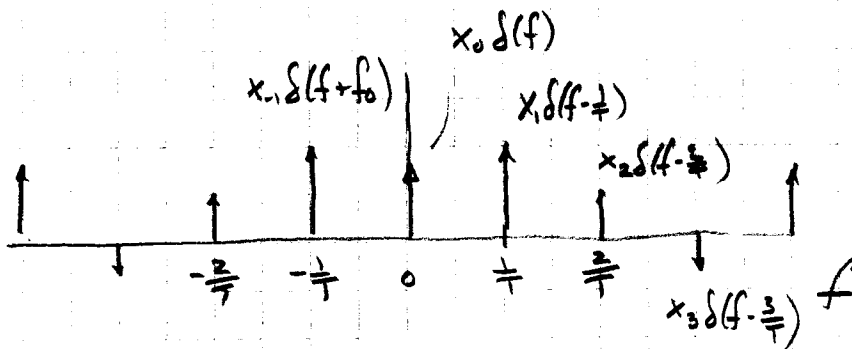
• What is the F transform of a periodic signal?



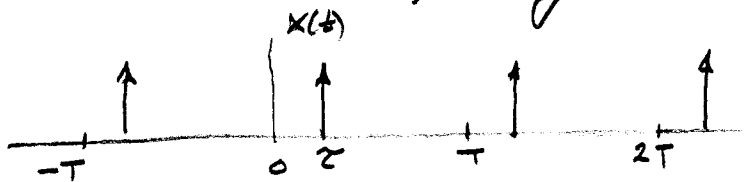
Expand in F series: $x(t) = \sum_k X_k e^{j2\pi kt/T}$

$$\text{Then } \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$= \sum_k X_k \delta(f - k/T) \quad \text{since } \mathcal{F}^{-1} \text{ gives } x(t)$$



• What is F transform of an impulse train?



$$x(t) = \sum_k \delta(t - kT - \tau)$$

A tricky question!

- First, $x(t)$ is periodic, so it has F series

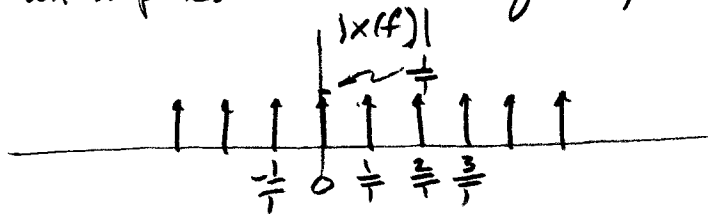
$$\begin{aligned} x(t) &= \sum_i X_i e^{j2\pi it/T} \\ &= \frac{1}{T} \sum_i e^{j2\pi i \frac{\tau}{T}} e^{j2\pi it/T} \\ &= \frac{1}{T} \sum_i e^{j2\pi i (t - \tau)/T} \end{aligned}$$

$$\begin{aligned} X_i &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi it/T} dt \\ &= \end{aligned}$$

- Next, to get the transform, we note that each complex exponential in the F series transforms to an offset impulse: $e^{j2\pi i t/T} \longleftrightarrow \delta(f - \frac{i}{T})$

- Then
$$X(f) = \frac{1}{T} \sum_i e^{j2\pi i z/T} \delta(f - i/T)$$

an impulse train in frequency!



If delay $z=0$, then $x(t) = \sum_k \delta(t - kT) \longleftrightarrow X(f) = \frac{1}{T} \sum_i \delta(f - i/T)$

so $\left\{ \begin{array}{l} \text{periodic} \\ \text{discrete} \end{array} \right\}$ in time, hence $\left\{ \begin{array}{l} \text{discrete} \\ \text{periodic} \end{array} \right\}$ in freq.