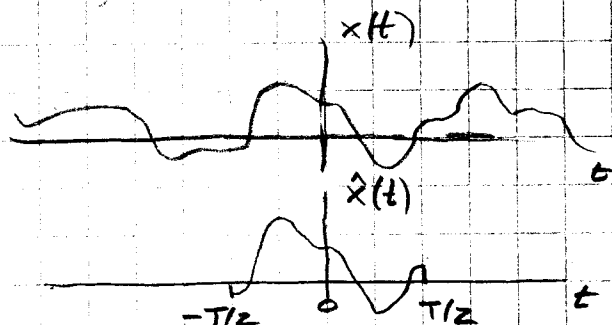


4.7 Windowing

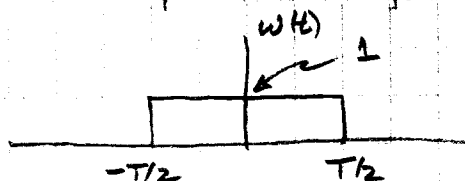
4.7.1

• Windowing of time records.

Suppose we would like the F transform of a signal, but only have a finite record length to work with. What is the relation between the true transform and that obtained from the record?



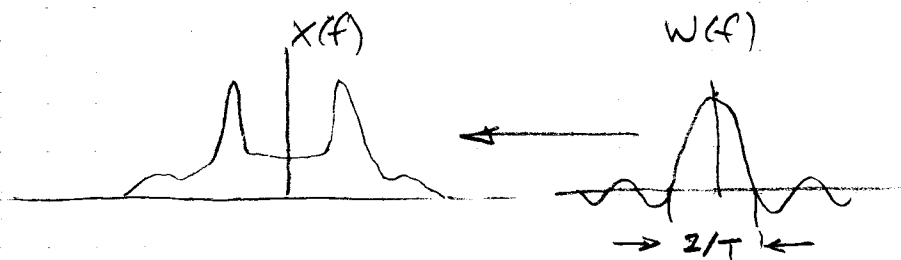
View it as multiplication of $x(t)$ by a rectangular window



$$\hat{x}(t) = x(t) w(t)$$

4.7.7

Then $\hat{X}(f) =$



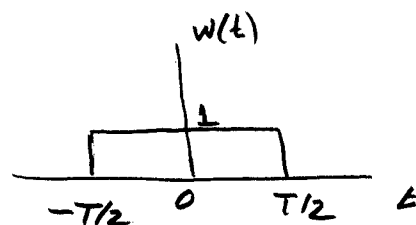
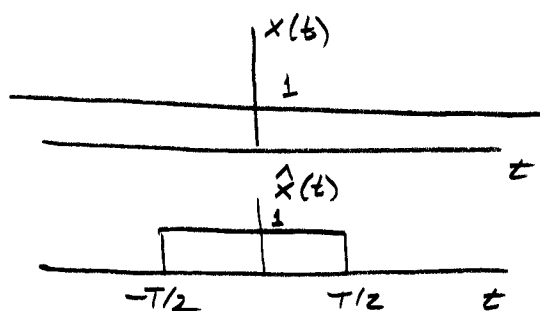
The convolution of $W(f)$ against $X(f)$ spreads the result in frequency, resulting in blurring, loss of frequency resolution

A longer record is better



for large T

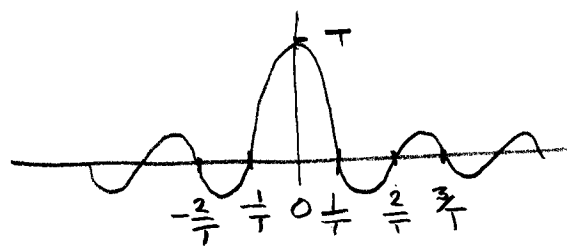
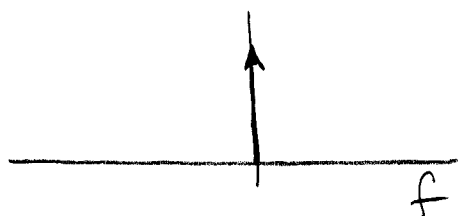
- Check this idea against a simple example:
→ window a dc signal



We already know the transform $\hat{X}(f) = T \operatorname{sinc}(fT)$.
Will this convolution approach give the same result?

$$X(f) = \delta(f)$$

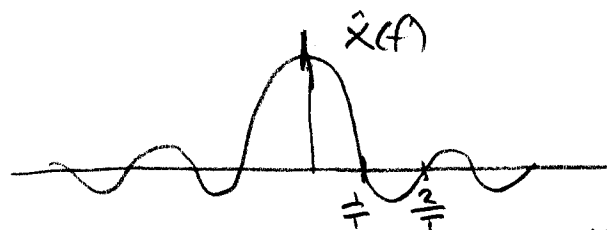
$$W(f) = T \operatorname{sinc}(fT)$$



$$\text{We try } \hat{X}(f) = X(f) \otimes W(f)$$

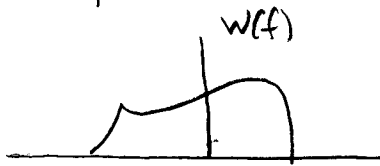
$$= \int_{-\infty}^{\infty} X(\eta - f) W(\eta) d\eta = \int_{-\infty}^{\infty} \delta(f - \eta) W(\eta) d\eta$$

$$= W(f) = T \operatorname{sinc}(fT) \quad \text{It works.}$$

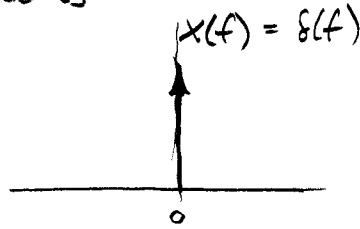


"spectral broadening"

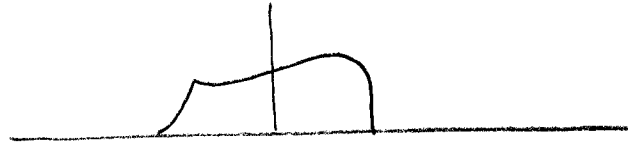
→ Note that convolving a function with an impulse reproduces the functions



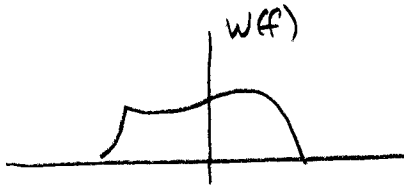
$\otimes$



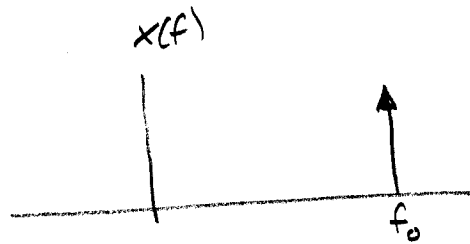
=



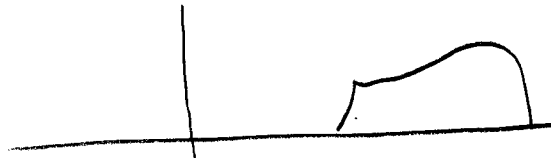
→ Convolution against a shifted unit impulse shifts the function.



$\otimes$



=



Q: Why is there an infinite range of frequencies required to represent dc (windowed dc) ?

A: Fourier analysis works with "eternal exponentials".

The original dc signal is one of them, so all we need is one component, so $\delta(f)$

But... the windowed pulse is created by the beating action of all those exponentials when they are summed (integrated).

$$\hat{x}(t) = \int_{-\infty}^{\infty} \hat{X}(f) e^{j2\pi ft} df$$

Test the notion that beating among "eternal exponentials" can produce windowed dc (i.e. a rectangular pulse). We'll approximate

$$\hat{x}_{\text{hat}}(t) = \int_{-\infty}^{\infty} T \cdot \text{sinc}(f \cdot T) \cdot e^{j2\pi f t} df$$

in two ways: first, approximate the integral as a sum over strips separated by Δf ; second, truncate the limits. For convenience, set $T := 1$ and define

$$K := 16 \quad 16 \text{ strips in } 1/T \text{ (an arbitrary choice), so that } \Delta f := \frac{1}{16} \cdot \frac{1}{T}$$

$$\text{sinc}(y) := \begin{cases} 1 & y=0 \\ \frac{\sin(\pi \cdot y)}{\pi \cdot y} & \text{otherwise} \end{cases}$$

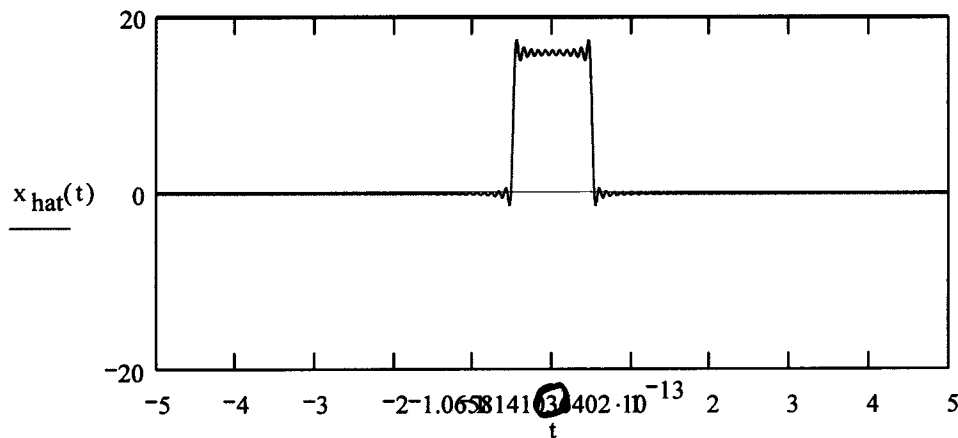
and sum over $2C+1$ ripples of the sinc function $C := 5$

so that the number of integration strips is $N := (2 \cdot C + 1) \cdot K$ and the approximation is

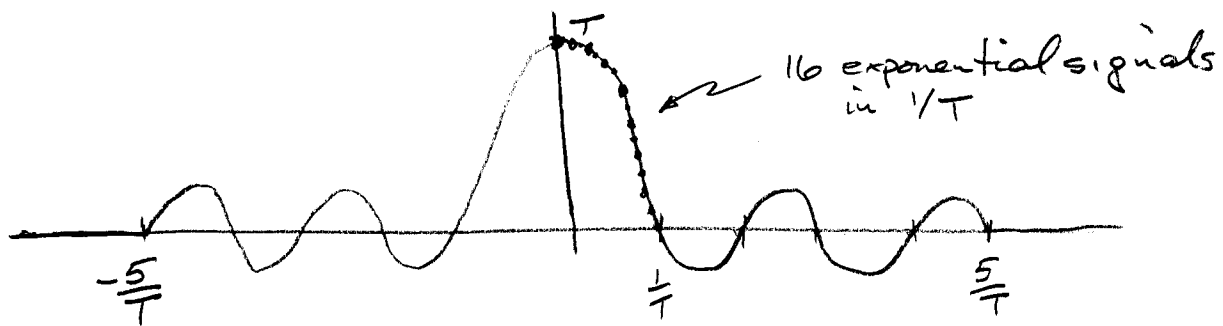
$$\hat{x}_{\text{hat}}(t) := \sum_{k=-N}^N T \cdot \text{sinc}(k \cdot \Delta f \cdot T) \cdot e^{j2\pi k \Delta f t}$$

And plot it

$t := -5, -4.99 \dots 5$

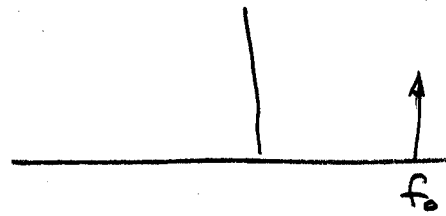
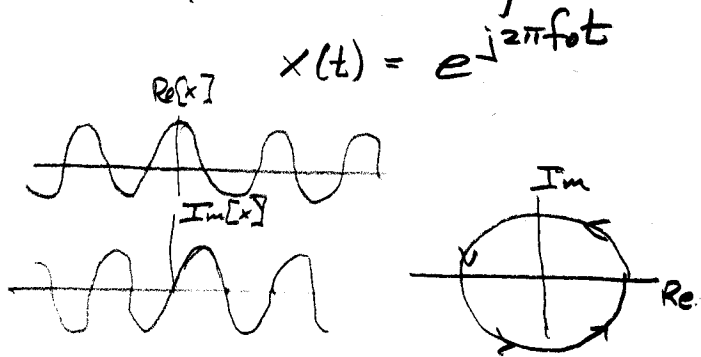


This is just the sum of complex exponential signals



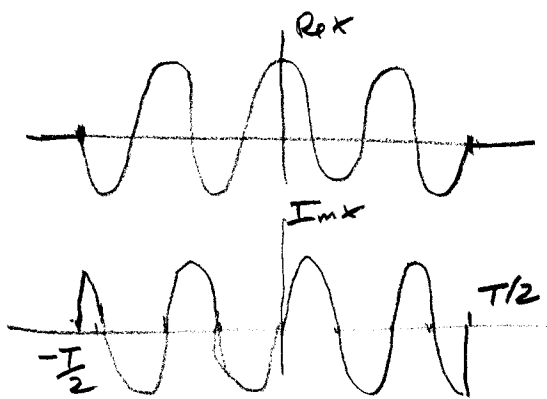
They interfere constructively for $|t| < \frac{1}{2}$,
destructively elsewhere.

- Try windowing a complex exponential:

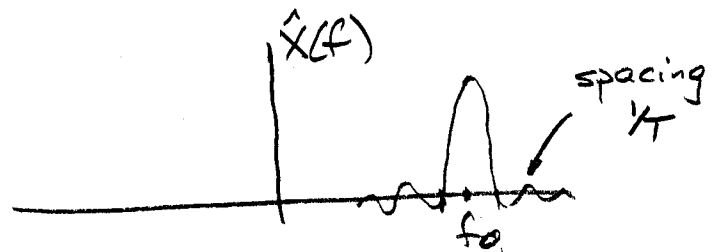


This is the "eternal" unwindowed version.

Now window it $\hat{x}(t) = x(t) w(t)$



$$\hat{X}(f) = X(f) \otimes W(f)$$



Spectral broadening again.

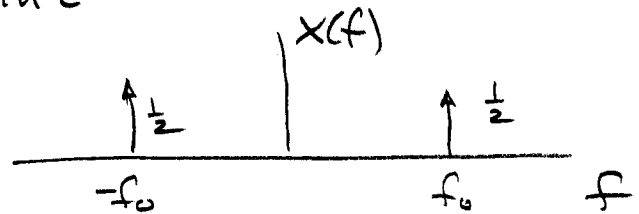
$$X(f) = \int \delta(\eta - f + f_0) W(\eta) d\eta = W(f - f_0)$$

Again, a wide range of frequencies (infinite range) is needed to produce a windowed tone.

- Same for cosine, sine

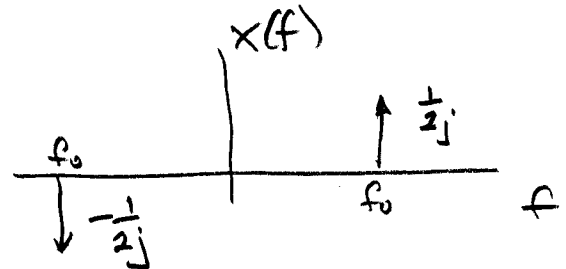
$$x(t) = \cos(2\pi f_0 t)$$

$$= \frac{1}{2} [e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}]$$

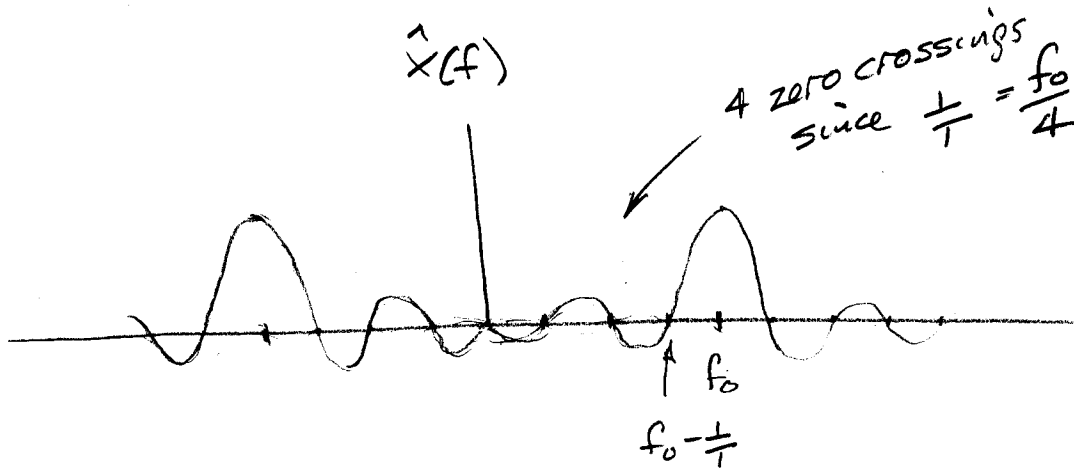
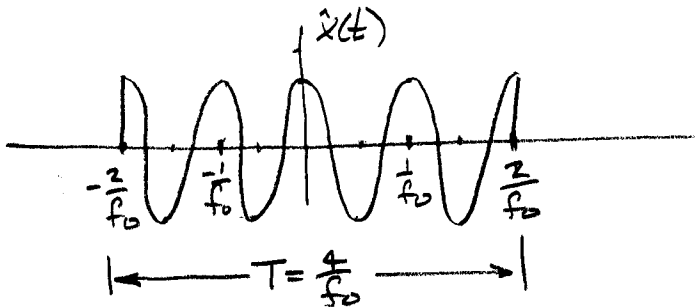


$$x(t) = \sin(2\pi f_0 t)$$

$$= \frac{1}{2j} [e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}]$$



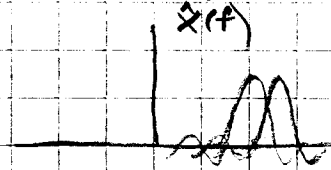
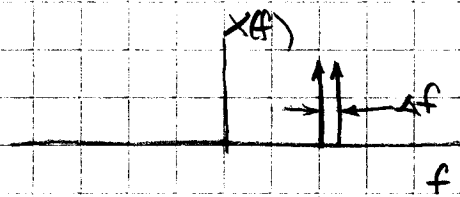
Truncated (windowed) to 4 cycles ($T = 4/f_0$)



Note we could have obtained this by direct integration:

$$\hat{X}(f) = \int_{-T/2}^{T/2} \cos(2\pi f_0 t) e^{-j2\pi f t} dt = \int_{-T/2}^{T/2} \cos(2\pi f_0 t) \cos(2\pi f t) dt$$

Can we resolve two sine waves with a finite record?



classical:
need $\neq \leq \Delta f$

$$T \geq \frac{1}{\Delta f}$$

To resolve tones spaced by 1 Hz, we need at least 1 sec of signal, regardless of tone frequencies.

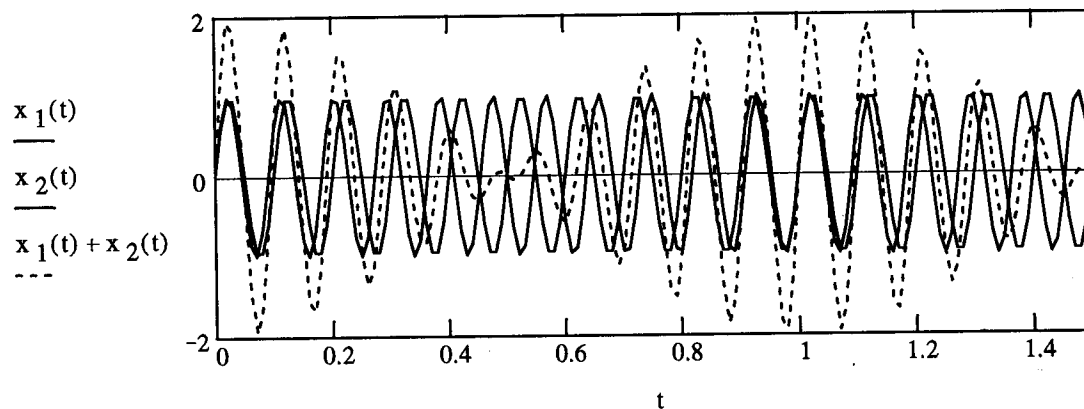
As usual, a parallel explanation in time domain:

$$f_1 := 10 \cdot \text{Hz}$$

$$f_2 := 11 \cdot \text{Hz}$$

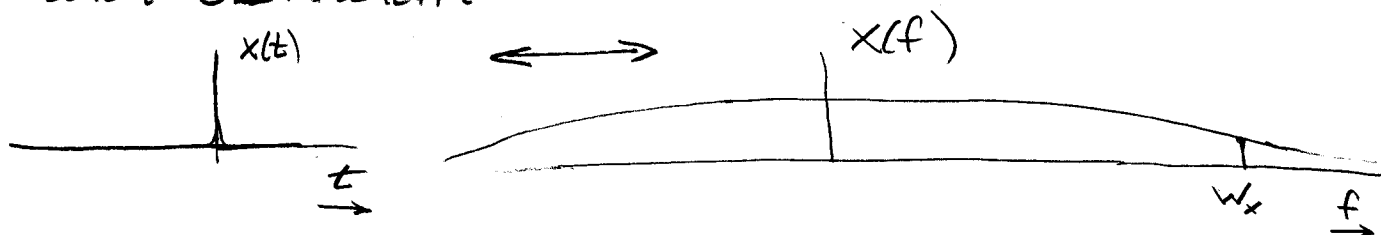
$$x_1(t) := \sin(2 \cdot \pi \cdot f_1 \cdot t)$$

$$x_2(t) := \sin(2 \cdot \pi \cdot f_2 \cdot t) \quad t := 0 \cdot \text{sec}, 0.01 \cdot \text{sec} \dots 1.5 \cdot \text{sec}$$

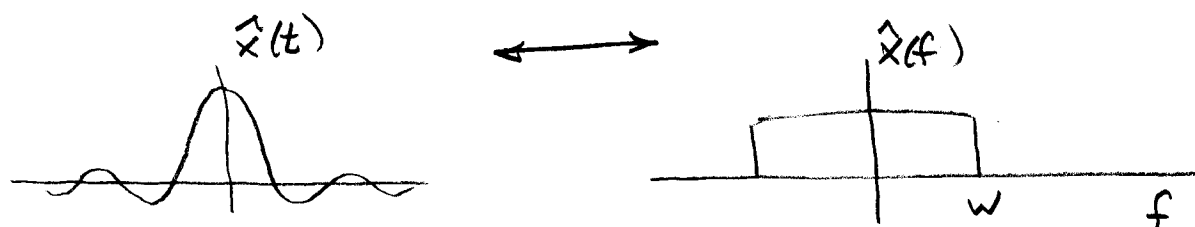


- By duality, these windowing arguments also apply in the frequency domain.

- Consider a very short pulse $x(t)$ with a correspondingly wide bandwidth



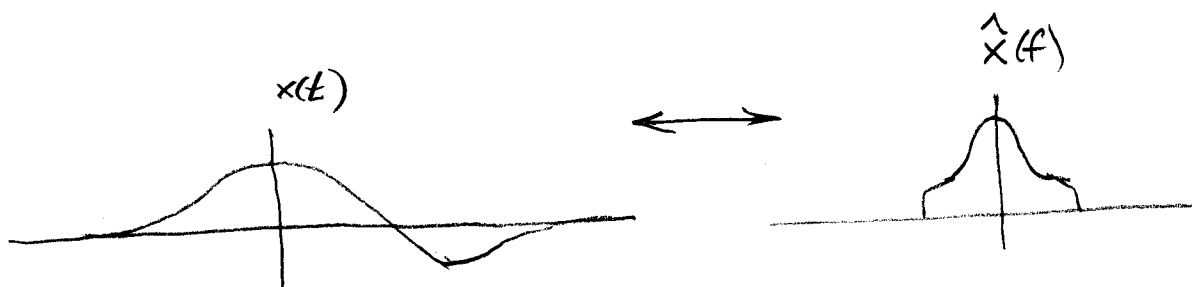
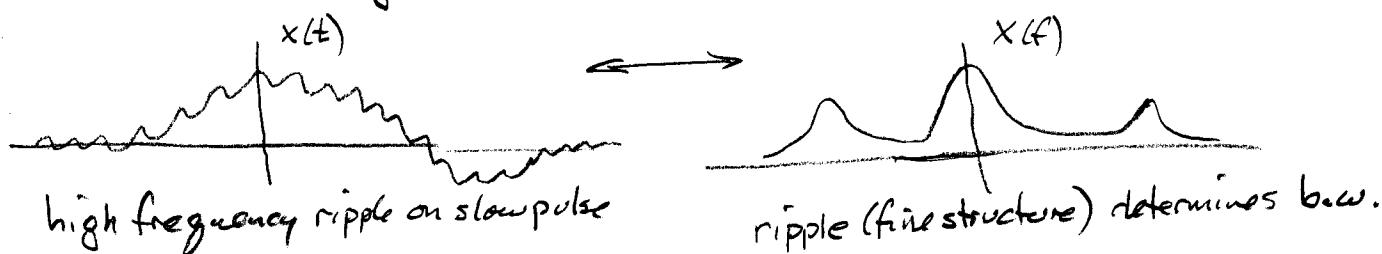
- Limit its frequency extent (i.e. b.w) with an ideal lowpass



slow changes, fine structure size increases

since eliminated the high frequencies

- Consider a signal $x(t)$ with both fine and coarse structure

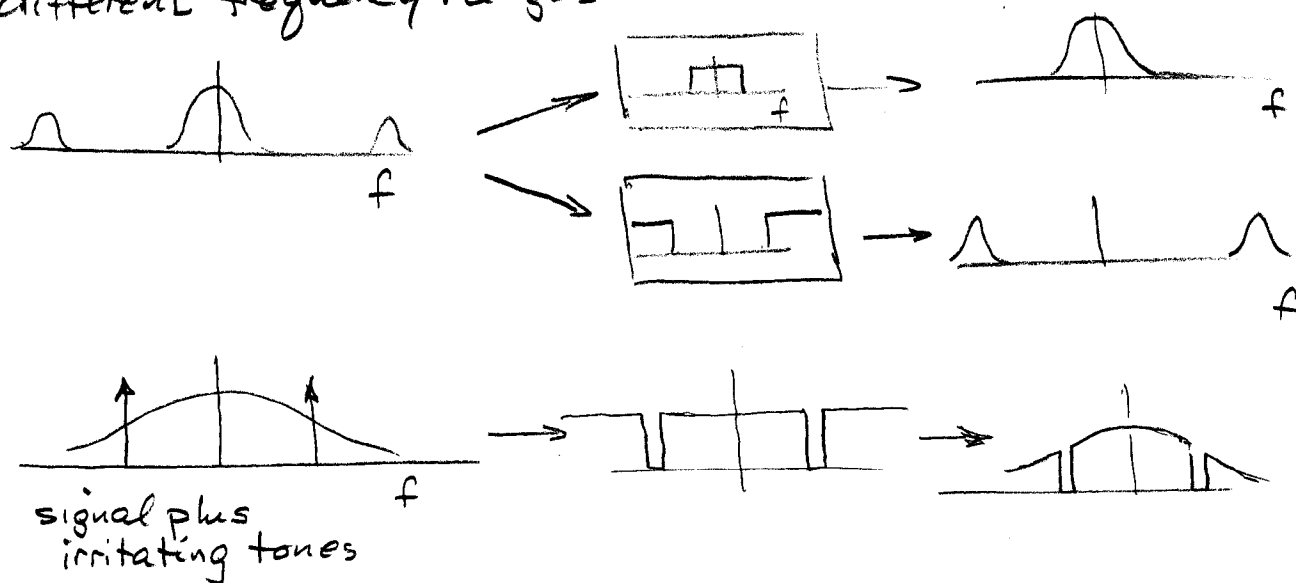


- In both examples, we see that lowpassing (windowing in freq domain) suppresses fine structure in time, thereby blurring it and reducing resolution.

But if the fine structure is noise, this is welcome!

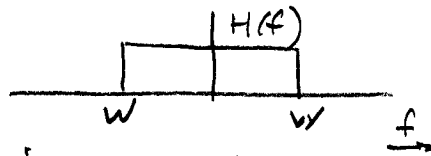
- Can we build an ideal lowpass filter?

- We often want to separate signals that occupy different frequency ranges

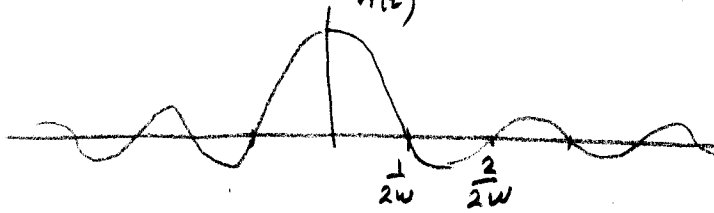


Consider ideal lowpass as the generic problem.

— We want



so we must implement impulse response $h(t)$



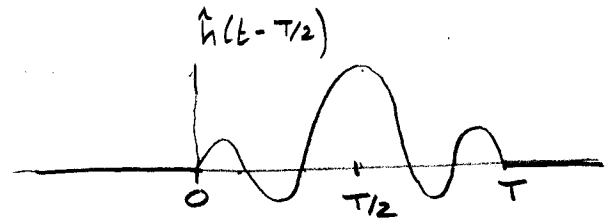
Two problems: — non causal
— infinite duration] so it's not possible

— But we really want it, so we're willing to accept an approximation:

— truncate it
(window it)



— delay it, so causal.



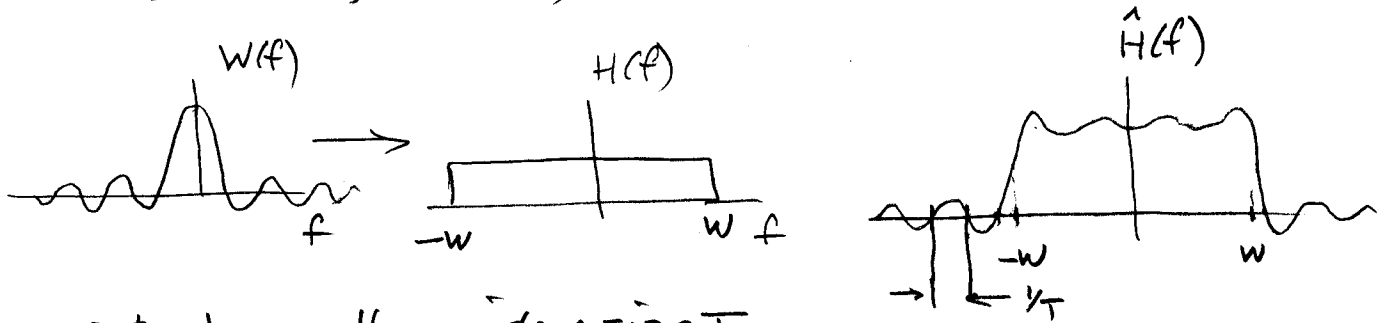
— Does it still look like an ideal lowpass?

$$\text{Freq resp} = \mathcal{F}[\hat{h}(t - T/2)] = e^{-j\pi f T} \mathcal{F}[\hat{h}(t)]$$

with respect to magnitude of the response, we can consider the noncausal $\hat{h}(t)$:

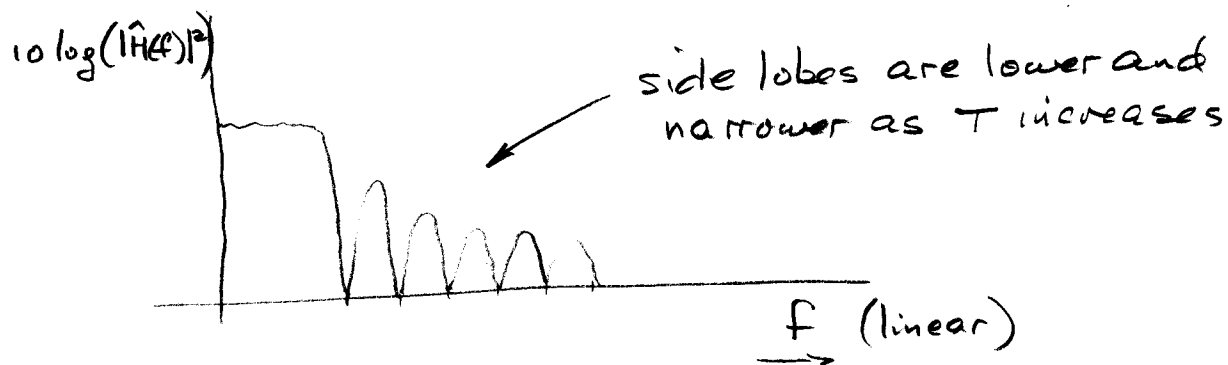
$$\hat{h}(t) = w(t) \cdot h(t) = w(t) 2W \text{sinc}(2Wt)$$

$$\hat{H}(f) = W(f) \otimes H(f)$$



The larger the window size T ,
the better the approximation.

Often presented with a dB scale:



Let's do a quick numerical investigation of the effect of windowing on the ideal lowpass.

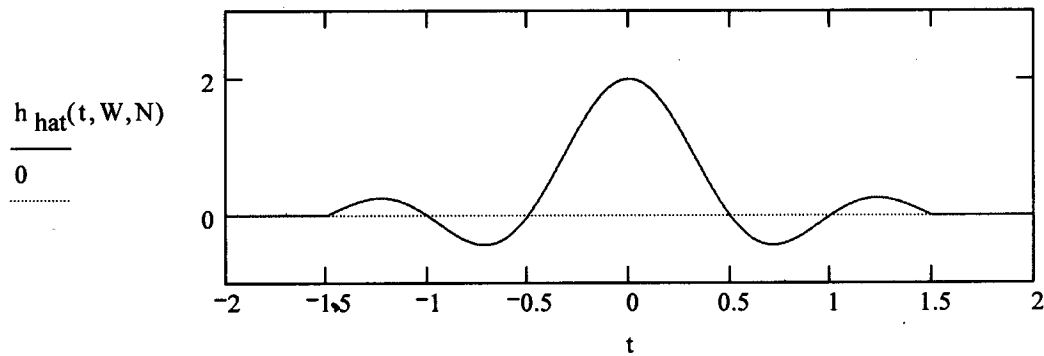
First define the sinc function: $\text{sinc}(x) := \begin{cases} 1 & x=0 \\ \frac{\sin(\pi \cdot x)}{\pi \cdot x} & \text{otherwise} \end{cases}$

so that the ideal lowpass impulse response is $h(t, W) := 2 \cdot W \cdot \text{sinc}(2 \cdot W \cdot t)$

Truncate it to N oscillations on either side, so that the window size is $T = 2 \cdot N \cdot \frac{1}{2 \cdot W}$

$$h_{\text{hat}}(t, W, N) := \left(\left| t \right| \leq \frac{N}{2 \cdot W} \right) \cdot h(t, W)$$

Plot an example: $N := 3$ $W := 1$ $t := -\frac{N+1}{2 \cdot W}, -0.99 \cdot \frac{N+1}{2 \cdot W} .. \frac{N+1}{2 \cdot W}$

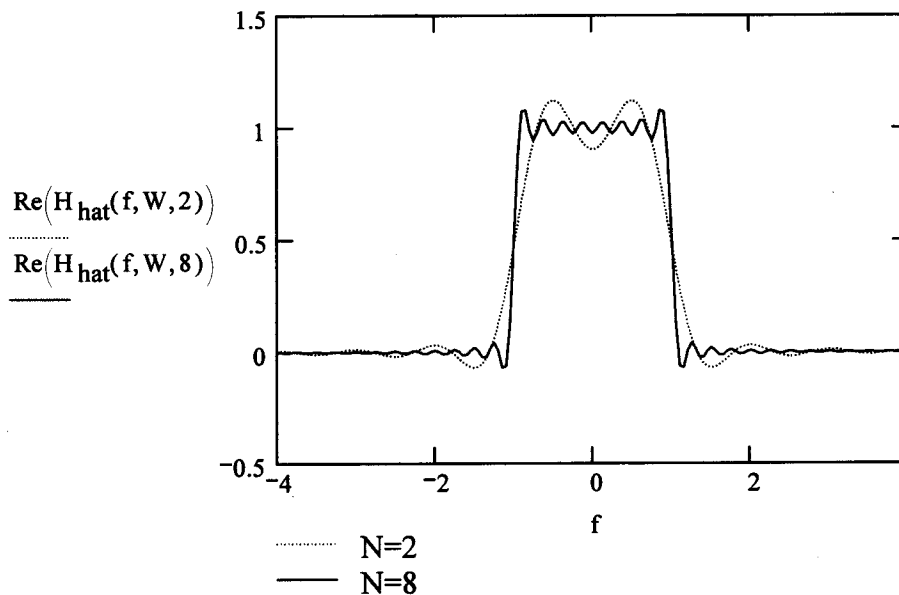


That was the time domain. Now we should see what the effect on the frequency response is. Here's how it's defined:

$$H_{\text{hat}}(f, W, N) := \int_{-\frac{N+1}{2 \cdot W}}^{\frac{N+1}{2 \cdot W}} h_{\text{hat}}(t, W, N) \cdot e^{-j \cdot 2 \cdot \pi \cdot f \cdot t} dt \quad \text{(It's faster by FFT, but this is just an example)}$$

All we have to do now is plot it. Choose the range of frequencies:

$$f := -4 \cdot W, -3.95 \cdot W .. 4 \cdot W$$



Now use a dB scale for the gain. Define $\text{dB}(x) := \text{if}(x < 10^{-14}, -140, 10 \cdot \log(x))$

