4.9 Fourier Wrapup (mostly review)

- Several weeks of study have shown the importance of the complex exponentials

$$e^{j2\pi ft}$$

as a basis in time and in freq.

- F. synthesis expresses a function of time as a sum (integral) of complex exponentials at various freq's, each with its complex amplitude $X(f)$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df \quad \text{or, for periodic,} \quad x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{j2\pi \frac{k}{T} t}$$

If you use ω, it's

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad \text{and periodic} \quad x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{j2\pi \frac{k}{T} t}$$

- F. analysis lets you calculate the corresponding function of frequency (the complex amplitudes)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \quad \text{or} \quad X_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi \frac{k}{T} t}\, dt$$

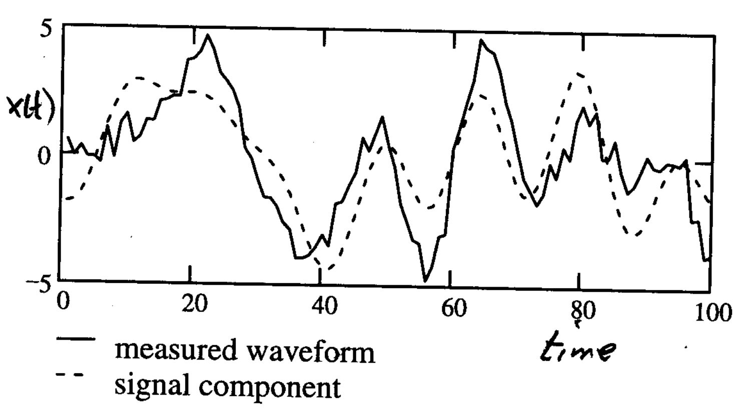
or $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

- You can view F analysis as expressing a f'n of freq as a sum (integral) of complex exponentials at various times, each with its weight $x(t)$

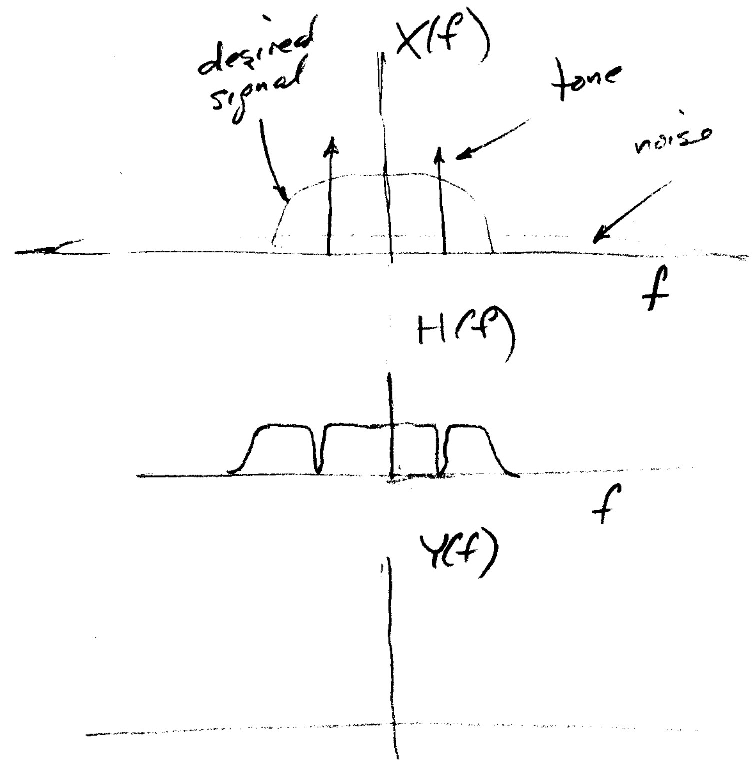
- In particular, $\mathcal{F}[h(t)] = H(f)$ is the freq response of a system with impulse resp $h(t)$
- If $y(t) = x(t) \circledast h(t)$ then $Y(f) = X(f)H(f)$

  Simplifies analysis of linear systems, or any situation involving convolution.
- If $y(t) = x(t)\,w(t)$ then $Y(f) = X(f) \circledast W(f)$

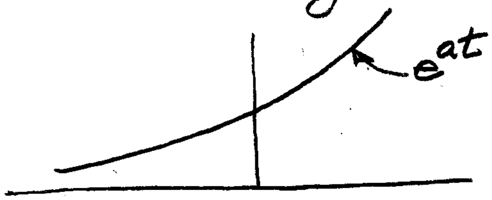
  A way to understand effects of windowing, modulation, etc on frequency makeup of a signal
- Thinking in both domains lets us understand signals better
- Thinking in both domains lets us design linear processing. Example



signal in noise and tone interference

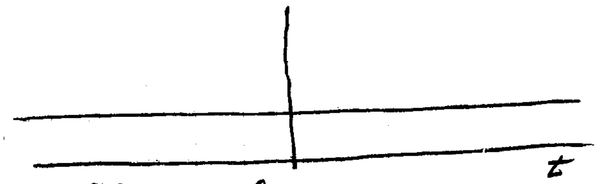


- Not all signals have a Fourier transform



$\int_{-\infty}^{\infty} e^{at} e^{-j2\pi ft} dt$ is undefined

Laplace transform helps here



$\int_{-\infty}^{\infty} 1\, e^{-j2\pi ft} dt$

similarly, but we resolve it by a notational trick called the delta function.

<u>Sufficient</u> conditions for existence of F transf. (Dirichlet)

– $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$ absolutely integrable

<u>and</u> – finite number of discontinuities, finite number maximama and minima in every time interval

- A useful distinction:

energy signal: $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

bounded, finite duration signals are energy signals

power signals $0 < \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$

eg. eternal exponentials, infinite duration signals such as noise or my lectures

Specific Properties

① A convolution in $\left\{ \begin{matrix} \text{time} \\ \text{freq} \end{matrix} \right\}$ transforms to a multiplication in $\left\{ \begin{matrix} \text{freq} \\ \text{time} \end{matrix} \right\}$

② It's corollary: $\xrightarrow[U(f)]{u(t)} \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \xrightarrow[Y(f)]{y(t)}$

$$y(t) = u(t) \circledast h(t) \qquad Y(f) = U(f)\,H(f)$$

③ Compression in $\left\{ \begin{matrix} \text{time} \\ \text{freq} \end{matrix} \right\}$ transforms to dilation in $\left\{ \begin{matrix} \text{freq} \\ \text{time} \end{matrix} \right\}$

④ Fine structure size in $\left\{ \begin{matrix} \text{time} \\ \text{freq} \end{matrix} \right\}$ is approx reciprocal of $\left\{ \begin{matrix} \text{freq span} \\ \text{duration} \end{matrix} \right\}$

⑤ Periodic signal in $\left\{ \begin{matrix} \text{time} \\ \text{freq} \end{matrix} \right\}$ transforms to discrete $\left\{ \begin{matrix} \text{spectrum} \\ \text{time function} \end{matrix} \right\}$

⑥ Specific transform pairs:

t	f
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\text{rect}_T(t)$	$T \,\text{sinc}\, fT$
$2B\,\text{sinc}(2Bt)$	$\text{rect}_{2B}(f)$
$\dot{x}(t)$	$j2\pi f\, X(f)$

t	f
$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f+f_0) + \frac{1}{2}\delta(f-f_0)$
$\sin 2\pi f_0 t$	$\frac{j}{2}\delta(f+f_0) - \frac{j}{2}\delta(f-f_0)$
periodic $x(t)$	$\sum_k X_k\, \delta(f - k/T)$
discrete $x(t)$	$\sum_i x(iT)\, e^{-j2\pi f i T}$

⑦ real signals have conjugate symmetric spectra

$$\text{real } x(t) \longleftrightarrow X(f), \quad X(-f) = X^*(f)$$

⑧ Parseval's theorem — same energy in both domains

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$ energy signals

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$ periodic signals (same power, or energy per cycle)

⑨ System characterizations:

$g(t)$ $h(t)$ $H(f)$