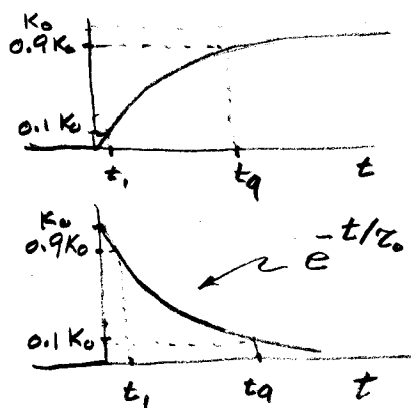
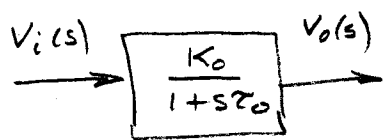


6.5 Proportional-Integral Control

6.5.1

- We saw that simple feedback around a first order lowpass leaves a non-zero steady state error. Increasing the gain enough to make that error small makes the drive signal unrealistically large if the reference signal jumps rapidly.
- We also saw that an integrator in the loop eliminates steady state error (for step inputs), but simple feedback doesn't allow simultaneous setting of bandwidth and damping factor.
- Combine these ideas with PI control for first order systems.
- The open loop system



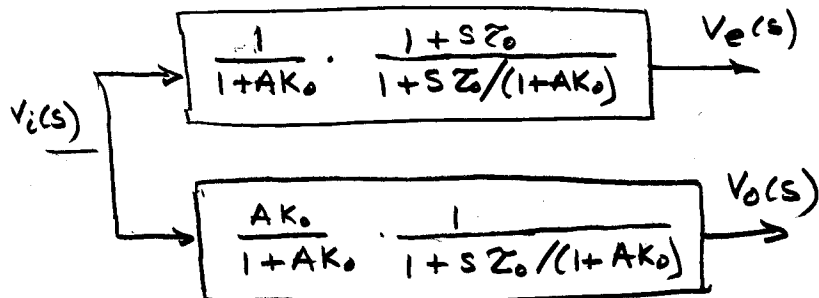
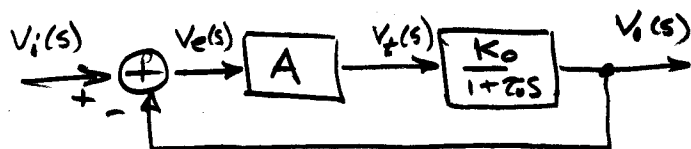
"rise time" of step response often defined as time from 10% to 90% of final value.

$$e^{-t_1/\tau_0} = 0.9 \quad t_1 = -\tau_0 \ln(0.9)$$

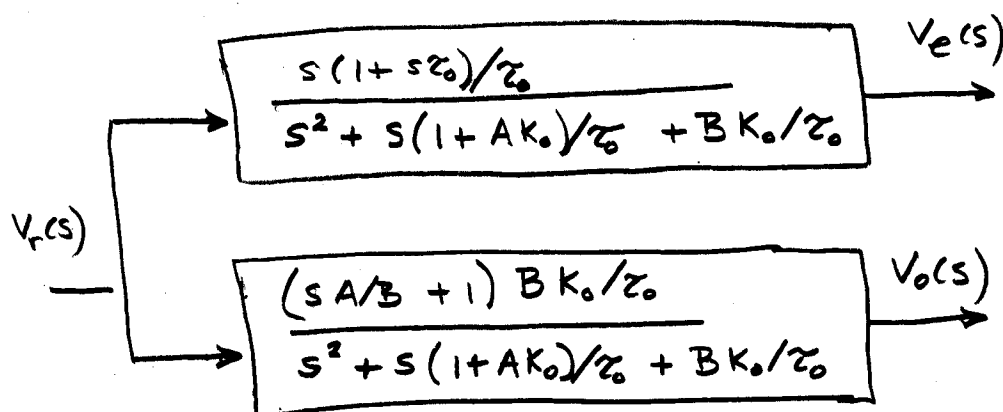
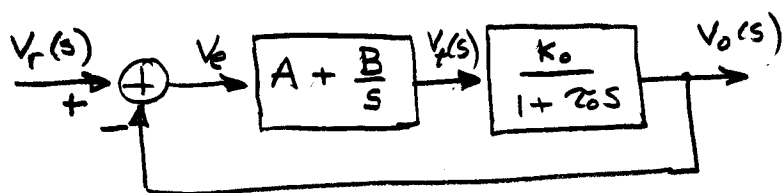
$$e^{-t_9/\tau_0} = 0.1 \quad t_9 = -\tau_0 \ln(0.1)$$

$$t_{rise} = t_9 - t_1 = \tau_0 \ln(9) = 2.2 \tau_0$$

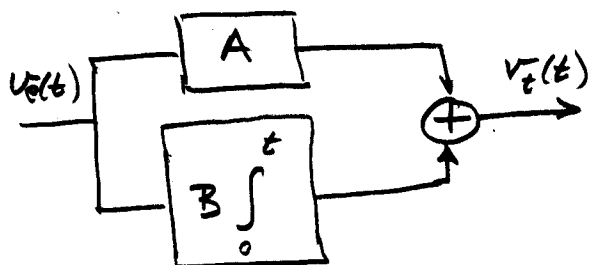
• Simple feedback (proportional control)



• PI control



The integrator branch can hold the appropriate steady state drive voltage, even if its input (the error) is zero.



- We have significantly modified the properties:

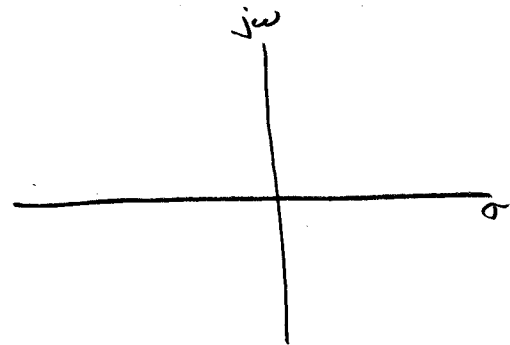
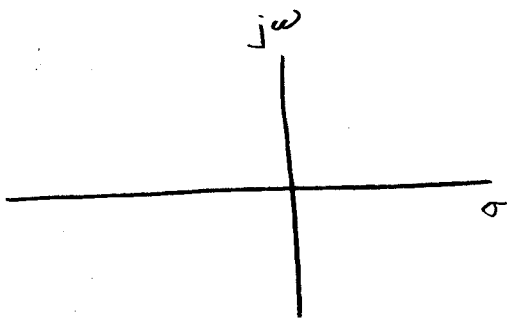
- if $v_r(t)$ is a unit step, then $v_e(\infty) = 0$
- if $v_r(t)$ is a unit ramp, then $v_e(\infty) = \frac{1}{BK_0}$
- it has become a second order system with a zero

- Poles $s^2 + 2\zeta\omega_0 s + \omega_0^2 = s^2 + s(1 + AK_0/\tau_0) + BK_0/\tau_0$

$$\omega_0 = \sqrt{BK_0/\tau_0} \quad \zeta = \frac{1}{2} \cdot \frac{1 + AK_0}{\sqrt{BK_0\tau_0}}$$

- Zeros $H_e(s) = 0$ at $s = 0, -1/\tau_0$

$$H_o(s) = 0 \text{ at } s = -B/A$$



We have control over the pole locations!

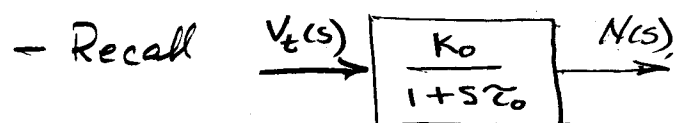
– choose B for ω_0

– choose A for ζ

but these also dictate the zero location

- Stability: Need $\zeta > 0 \Rightarrow A > -1/K_0$

- Example Speed control of a dc motor with inertial, frictional load



$$K_o = \frac{K_a \Phi}{R_a b + (K_a \Phi)^2} \quad \tau_o = \frac{R_a J}{R_a b + (K_a \Phi)^2}$$

where

$$K_a \Phi = 2 \frac{\text{volt}}{\text{rad/sec}}$$

$$R_a = 0.5 \Omega$$

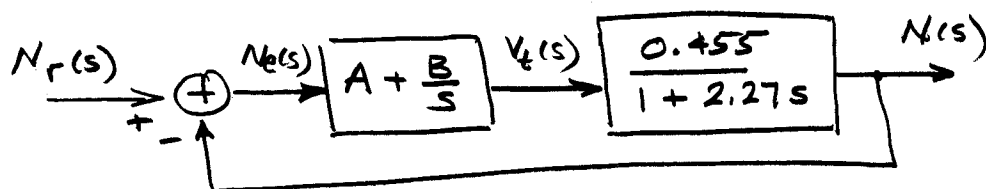
$$b = 0.8 \frac{\text{nt} \cdot \text{m}}{\text{rad/sec}}$$

$$J = 20 \text{ kg} \cdot \text{m}^2$$

so $K_o = 0.455 \text{ rad/sec} \cdot \text{volt}$

$$\tau_o = 2.27 \text{ sec}$$

$$t_{\text{rise}} = 2.2 \tau_o = 5 \text{ sec}$$



- General: choose A, B to improve response time as well as eliminate steady state error.

Aim for $\zeta = \frac{1}{\sqrt{2}}$, fast with small overshoot.

Aim for $\omega_o > \frac{1}{\tau_o}$ to make response faster.

The system has a zero, too, so response differs from that of a 2nd order low pass

- Design approaches:

- invert the closed loop transform as a function of A, B ; then try various A, B until it looks good.

or - reduce number of parameters by reasonable choices, then invert. That's what we'll do.

- Assume new b/w ω_0 is larger than open loop b/w $1/\tau_0$ (both in rad/sec).

$$\omega_0 = \frac{\Omega}{\tau_0} \quad \omega_0 \tau_0 = \Omega$$

Then from p. 6.5.3
$$B = \frac{\Omega^2}{K_0 \tau_0}$$

$$\text{Set } \xi = \frac{1}{\sqrt{2}} = \frac{1}{2} \frac{1 + AK_0}{\sqrt{BK_0\tau_0}} = \frac{1 + AK_0}{2\Omega}$$

so
$$A = \frac{\sqrt{2}\Omega - 1}{K_0}$$

and the zero location is $-\frac{B}{A} = \frac{\Omega^2}{\sqrt{2}\Omega - 1} \frac{1}{\tau_0}$

Substitute these values into the closed loop transfer function p. 6.5.2

$$H_c(s) = \frac{\left(s \frac{\tau_0(\sqrt{2}\Omega - 1)}{\Omega^2} + 1\right) (\Omega/\tau_0)^2}{s^2 + s\sqrt{2}\Omega/\tau_0 + \Omega^2/\tau_0^2}$$

inversion
see
Appendix
p 6.5.12

$$h_c(t) = \frac{1}{\tau_0} e^{-\frac{\Omega t}{\sqrt{2}\tau_0}} \left((\sqrt{2}\Omega - 1) \cos\left(\frac{\Omega t}{\sqrt{2}\tau_0}\right) + \sin\left(\frac{\Omega t}{\sqrt{2}\tau_0}\right) \right)$$

$$g_c(t) = 1 - e^{-\frac{\Omega t}{\sqrt{2}\tau_0}} \left(\cos\left(\frac{\Omega t}{\sqrt{2}\tau_0}\right) - \left(1 - \frac{\sqrt{2}}{\Omega}\right) \sin\left(\frac{\Omega t}{\sqrt{2}\tau_0}\right) \right)$$

Proportional Integral Control of DC Motor Speed

The load is a combination of moment of inertia and friction. Motor and load constants:

$$K_a\Phi := 2 \cdot \text{volt} \cdot \frac{\text{sec}}{\text{rad}} \quad R_a := 0.5 \cdot \text{ohm}$$

$$b := 0.8 \cdot \text{newton} \cdot \text{m} \cdot \frac{\text{sec}}{\text{rad}} \quad J := 20 \cdot \text{kg} \cdot \text{m}^2$$

Derived parameters of a first order model: $H_o(s) = \frac{K_o}{1 + s \cdot \tau_o}$

$$K_o := \frac{K_a\Phi}{R_a \cdot b + K_a\Phi^2} \quad K_o = 0.455 \cdot \frac{\text{rad}}{\text{sec} \cdot \text{volt}}$$

$$\tau_o := \frac{R_a \cdot J}{R_a \cdot b + K_a\Phi^2} \quad \tau_o = 2.273 \text{ s} \quad \text{the time constant}$$

With PI control, the controller has transfer function $A + \frac{B}{s}$

Closed loop transfer functions

$$H_v(s) = \frac{\left(s \cdot \frac{A}{B} + 1\right) \cdot B \cdot \frac{K_o}{\tau_o}}{s^2 + s \cdot \left(\frac{1 + A \cdot K_o}{\tau_o}\right) + B \cdot \frac{K_o}{\tau_o}} \quad H_e(s) = \frac{s \cdot \left(s + \frac{1}{\tau_o}\right)}{s^2 + s \cdot \left(\frac{1 + A \cdot K_o}{\tau_o}\right) + B \cdot \frac{K_o}{\tau_o}}$$

Define standard parameters

$$\omega_o = B \cdot \frac{K_o}{\tau_o} \quad \zeta = \frac{1}{2} \cdot \frac{(1 + A \cdot K_o)}{\tau_o \cdot \omega_o}$$

Set the bandwidth expansion $\Omega = \omega_o \cdot \tau_o$ and damping factor $\zeta = \frac{1}{\sqrt{2}}$

Then the transfer function can be written

$$H_v(s) = \frac{\Omega^2}{\tau_o^2} \cdot \frac{\left[s \cdot \frac{(\sqrt{2} \cdot \Omega - 1)}{\Omega^2} \cdot \tau_o + 1 \right]}{\left(s^2 + s \cdot \sqrt{2} \cdot \frac{\Omega}{\tau_o} + \frac{\Omega^2}{\tau_o^2} \right)}$$

and by inversion, we get the impulse and step responses

$$h_v(t, \Omega) := \frac{1}{\tau_o} \cdot \exp\left(\frac{-\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \cdot \left[(\sqrt{2} \cdot \Omega - 1) \cdot \cos\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) + \sin\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \right]$$

$$g_v(t, \Omega) := 1 - \exp\left(\frac{-\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \cdot \left[\cos\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) - \left(1 - \frac{\sqrt{2}}{\Omega}\right) \cdot \sin\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \right]$$

The original open loop step response, normalized by K_o , is

$$g_o(t) := 1 - e^{\frac{-t}{\tau_o}}$$

and the step response of a second order *standard* lowpass, for comparison.

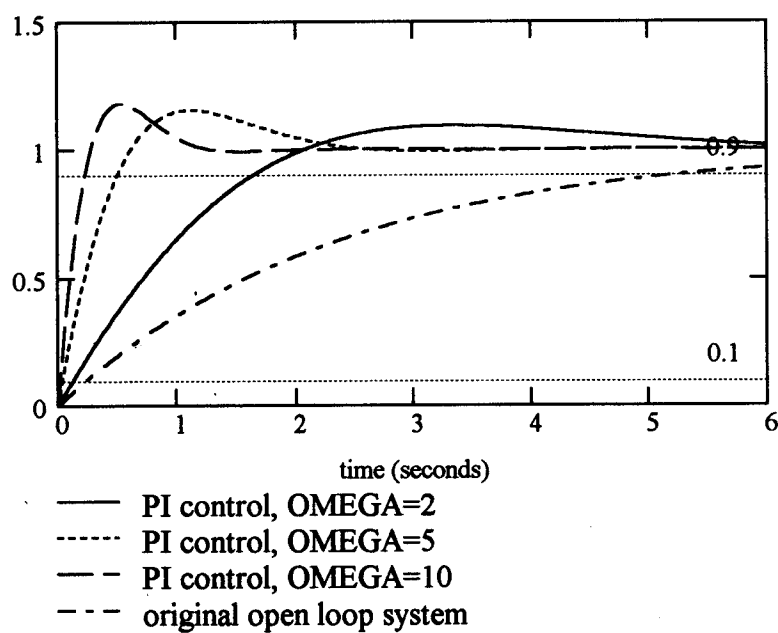
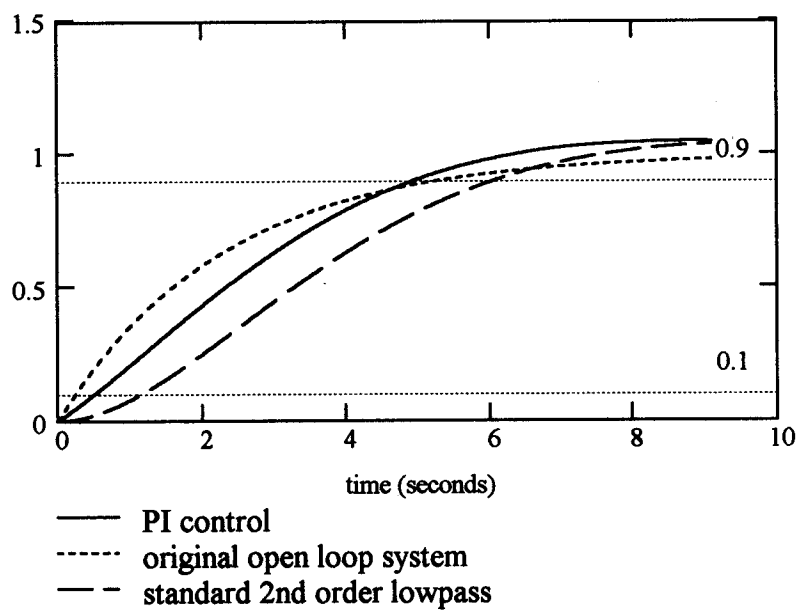
$$g_{lp}(t, \Omega) := 1 - \exp\left(\frac{-\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \cdot \left(\cos\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) + \sin\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \right)$$

Time axis for plots

$$t := 0 \cdot \tau_o, 0.01 \cdot \tau_o \dots 4 \cdot \tau_o$$

Comparison of the three types of response for bandwidth expansion $\Omega := 1$

Note rise time markers shown

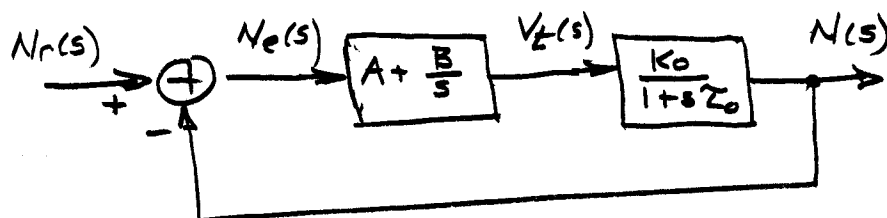


Note the rise time improvement as Ω increases.

- Evidently, PI control eliminates steady state error, keeps the overshoot oscillations manageable and improves speed of response

- Q: How does it achieve response time improvement? A: Increased drive voltage $v_t(t)$.
- Q: How big is the increase in $v_t(t)$? A: That's what we'll look at next.

- Recall system block diagram



We could:

- derive another transfer function $H_t(s) = V_t(s)/N(s)$ then multiply by $1/s$ and invert
- multiply the existing $H_v(s) = N(s)/N_r(s)$ by $(1 + s z_o)/K_o$, multiply by $1/s$ and invert
- or use the existing $g_v(t)$ and form

$$g_t(t) = \frac{1}{K_o} (g_v(t) + z_o \dot{g}_v(t))$$

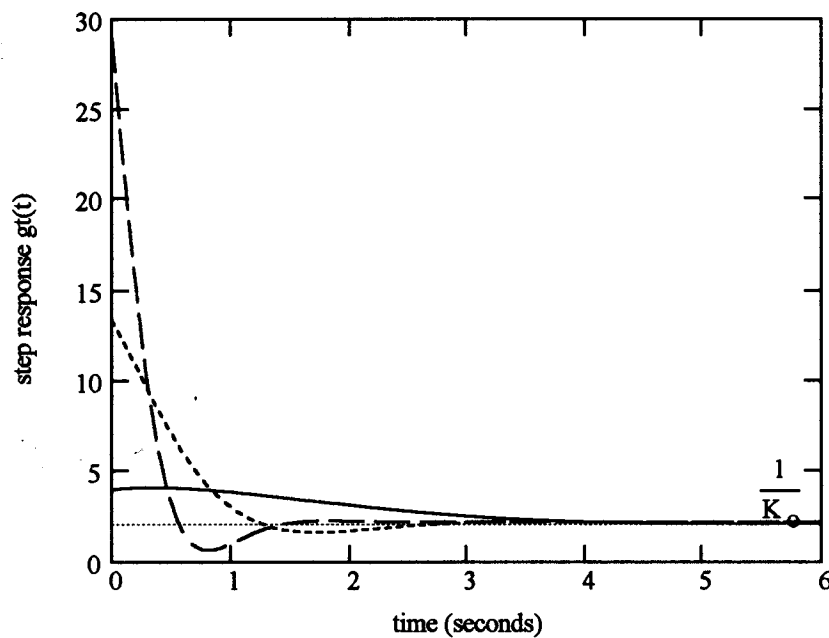
This seems easiest, on the whole.

$$g_v(t) = 1 - \exp\left(\frac{-\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \cdot \left[\cos\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) - \left(1 - \frac{\sqrt{2}}{\Omega}\right) \cdot \sin\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \right]$$

$$\tau_o \frac{d}{dt} g_v(t) = \exp\left(\frac{-\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \cdot \left[(\sqrt{2} \cdot \Omega - 1) \cdot \cos\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) + \sin\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \right]$$

$$g_t(t) = \frac{1}{K_o} \cdot \left(g_v(t) + \tau_o \frac{d}{dt} g_v(t) \right)$$

$$g_t(t, \Omega) := \frac{1}{K_o} \cdot \left[1 - \exp\left(\frac{-\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \cdot \left[(2 - \Omega \cdot \sqrt{2}) \cdot \cos\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) - \left(2 - \frac{\sqrt{2}}{\Omega}\right) \cdot \sin\left(\frac{\Omega}{\sqrt{2} \cdot \tau_o} \cdot t\right) \right] \right]$$



- PI control, OMEGA=2
- PI control, OMEGA=5
- - - PI control, OMEGA=10

Step Response of Terminal Voltage

- The response time improvement requires large terminal voltages. We can find the peak value by the IVT:

$$G_t(s) = \frac{1}{s} \cdot \frac{1+s\tau_0}{K_0} \cdot H_v(s)$$

$$= \frac{1}{s} \cdot \frac{1+s\tau_0}{K_0} \cdot \left(\frac{\Omega}{\tau_0}\right)^2 \frac{s\tau_0(\sqrt{2}\Omega-1) + 1}{s^2 + s\sqrt{2}\Omega/\tau_0 + \Omega/\tau_0^2}$$

$$q_t(\infty) = \lim_{s \rightarrow \infty} sG_t(s) = \left(\frac{\Omega}{\tau_0}\right)^2 \frac{\tau_0}{K_0} \cdot \frac{\tau_0(\sqrt{2}\Omega-1)}{\Omega^2}$$

$$q_t(0+) = \frac{\sqrt{2}\Omega-1}{K_0}$$

That is $\sqrt{2}\Omega-1$ times its final value, which shows explicitly the link with response time improvement

- To put it in perspective, though, we have zero steady state error and we can choose the speedup factor. In contrast, significant improvement in st.st. error in proportional control requires truly huge terminal voltages if the reference jumps suddenly.

Appendix : Inversion of $G_v(s)$.

$$G_v(s) = \frac{1}{s} H_v(s) = \frac{1}{s} \cdot \frac{\Omega^2}{\tau_0^2} \cdot \frac{s \frac{(\sqrt{2}\Omega - 1)\tau_0}{\Omega^2} + 1}{s^2 + s\sqrt{2} \frac{\Omega}{\tau_0} + \frac{\Omega^2}{\tau_0^2}}$$

Simplify notation to

$$G_v(s) = a^2 \frac{bs+1}{s(s^2 + \sqrt{2}as + a^2)} = a^2 \frac{bs+1}{s\left(s + \frac{a}{\sqrt{2}}(1+j)\right)\left(s + \frac{a}{\sqrt{2}}(1-j)\right)}$$

where $a = \frac{\Omega}{\tau_0}$ $b = \frac{(\sqrt{2}\Omega - 1)\tau_0}{\Omega^2}$

$$\text{Res}_0 [G_v(s)e^{st}] = \left(s G_v(s) e^{st} \right) \Big|_{s=0} = 1$$

$$\begin{aligned} \text{Res}_{-\frac{a}{\sqrt{2}}(1+j)} [G_v(s)e^{st}] &= \left(\left(s + \frac{a}{\sqrt{2}}(1+j) \right) G_v(s) e^{st} \right) \Big|_{s = -\frac{a}{\sqrt{2}}(1+j)} \\ &= \frac{a^2 \left(-\frac{ab}{\sqrt{2}}(1+j) + 1 \right) e^{-at/\sqrt{2}} e^{-j\frac{a}{\sqrt{2}}t}}{-\frac{a}{\sqrt{2}}(1+j)(-\sqrt{2}ja)} \end{aligned}$$

$$g_v(t) = \text{Res}_0 + 2 \text{Res}_{-\frac{a}{\sqrt{2}}(1+j)}$$

$$= 1 - e^{-\frac{a}{\sqrt{2}}t} \left(\cos\left(\frac{a}{\sqrt{2}}t\right) + (1 - \sqrt{2}ab) \sin\left(\frac{a}{\sqrt{2}}t\right) \right) \quad \text{after simplification}$$

substitute a, b back in

$$g_v(t) = 1 - e^{-\frac{\Omega}{\sqrt{2}\tau_0}t} \left(\cos\left(\frac{\Omega}{\sqrt{2}\tau_0}t\right) + \left(\frac{\sqrt{2}-\Omega}{\Omega}\right) \sin\left(\frac{\Omega}{\sqrt{2}\tau_0}t\right) \right)$$