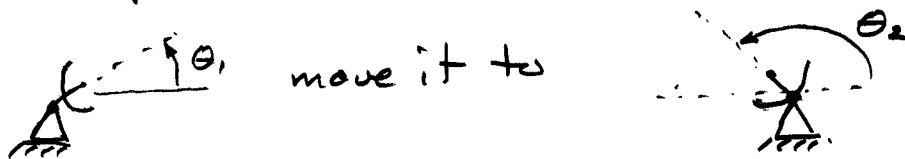


6.6 Elementary Nonlinearities in the Loop

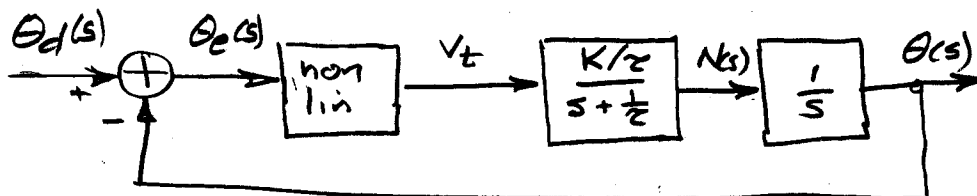
- We know that proportional control needs a large loop gain to reduce steady state error, which makes it either
 - dangerous
 - or unrealistic
 if the reference setting is inclined to step changes.

- For example, consider an antenna tracking system:



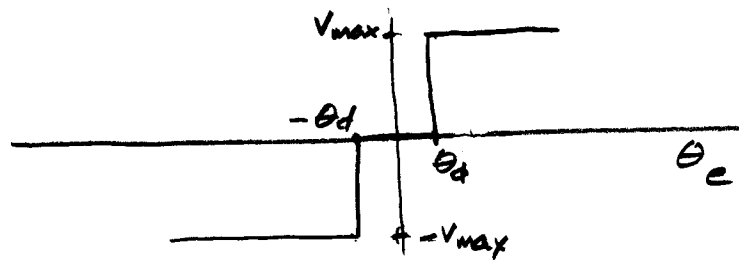
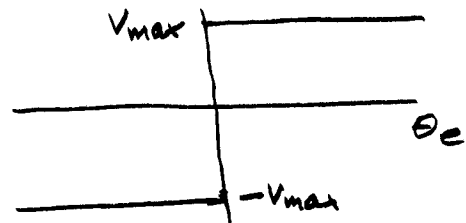
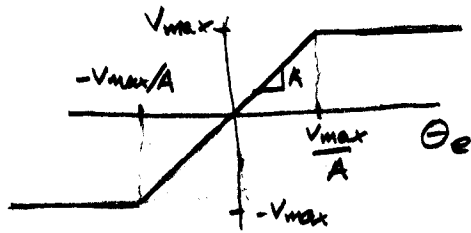
It's not reasonable to apply a huge voltage to the motor and expect a damped exponential trajectory.

It's more likely that we apply some maximum voltage to the motor, have it run a steady speed until it reaches its destination.



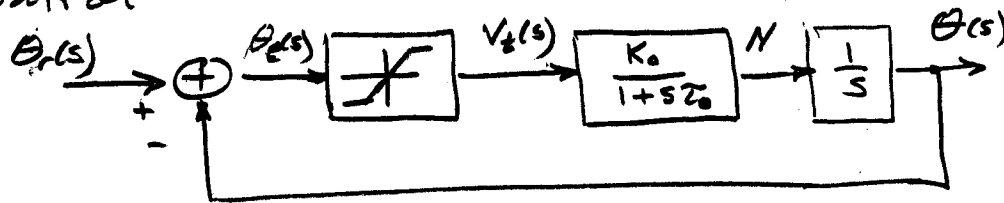
- Such nonlinearities are very common, and they represent the reality of saturation better than purely linear models.

Typical nonlinearities

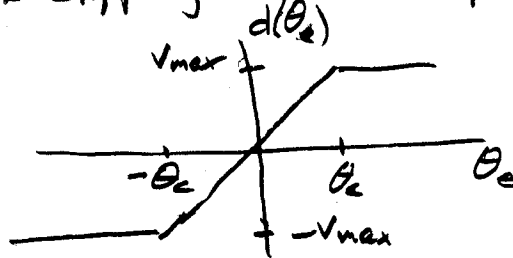


- To analyze these, we need a DE for each regime, with ICs inherited from the previous section of the trajectory. Each DE may be solved in the time or transform domain.

- For our example, we'll take the dc motor position control

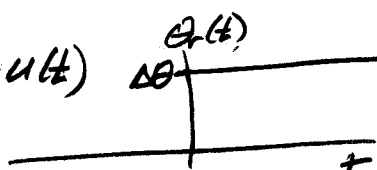


with a clipping nonlinearity $v_t = d(v_e)$

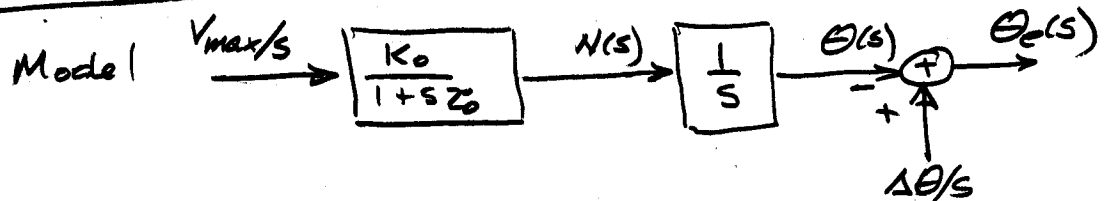


slope $A = V_{max}/\Theta_c$

- If starting from rest, a jump in Θ of Θ_c or less lets it behave linearly.
- If the jump is larger, then two regimes:
 - v_t constant at V_{max} until $|\Theta_r - \Theta| < \Theta_c$
 - then linear

Suppose $\Theta_r(t) = \Delta\Theta u(t)$  $\Delta\Theta > \Theta_c$

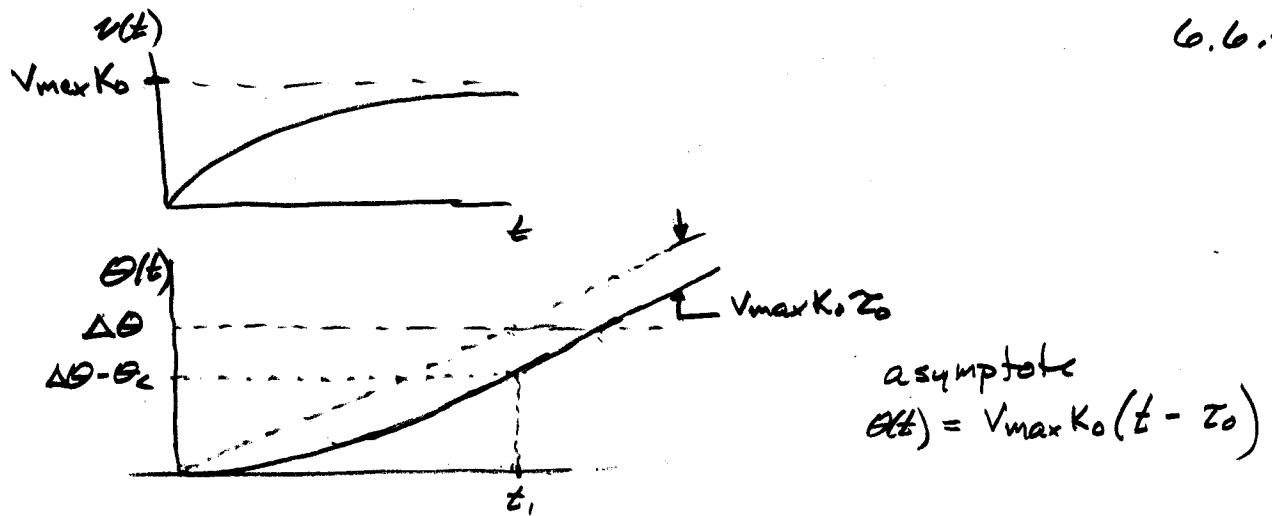
First regime



so DE or transform gives

$$v(t) = V_{max} K_o (1 - e^{-t/\tau_o})$$

$$\Theta(t) = \int_0^t v(\alpha) d\alpha = V_{max} K_o (t - \tau_o (1 - e^{-t/\tau_o}))$$



First regime ends at t_1 , when

$$\theta(t_1) = \Delta\theta - \theta_c \quad \text{or} \quad \theta_c(t) = \theta_c$$

We need t_1 , $\theta(t_1)$, $v(t_1)$ as boundary conditions for the next regime.

If we need an exact solution for t_1 , then solve $\Delta\theta - \theta_c = V_{\max} K_0 (t - z_0 (1 - e^{-t/z_0}))$ numerically.

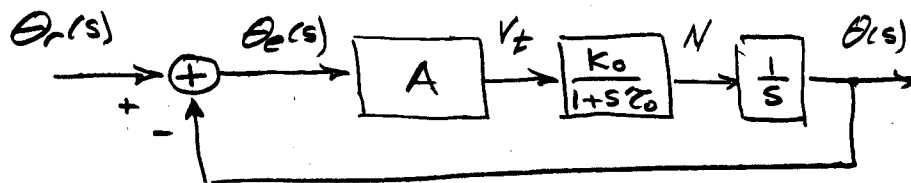
We'll use the asymptote, on the grounds that the shaft has reached maximum speed asymptotically, and therefore gives us maximum trouble with overshoot. Then end of first regime has

$$\theta_1 = \Delta\theta - \theta_c \quad t_1 \approx \frac{\Delta\theta - \theta_c}{V_{\max} K_0} + z_0$$

$$v_1 \approx V_{\max} K_0$$

Second regime

Here θ_e has been reduced to $< \theta_c$
 so we're in the proportional control regime.



Strategy: - collapse loop to get $H(s) = \theta(s)/\theta_r(s)$
 - convert to DE, then include ICs
 - solve the DE, either in time domain or by retransformation.

- get transfer function:

$$\frac{\theta_r(s)}{\theta(s)} = \frac{\frac{AK_0}{s(1+s\tau_0)}}{1 + \frac{AK_0}{s(1+s\tau_0)}} = \frac{AK_0/\tau_0}{s^2 + \frac{1}{\tau_0}s + AK_0/\tau_0}$$

- get the DE (for convenience, start at $t=0$, not $t=t_1$)

$$\ddot{\theta}(t) + \frac{1}{\tau_0} \dot{\theta}(t) + \frac{AK_0}{\tau_0} \theta(t) = \frac{AK_0}{\tau_0} \theta_r(t) \quad t \geq 0$$

with $\theta(0^-) = \Delta\theta - \theta_c$, $\dot{\theta}(0^-) = V_{max} K_0$

and $\theta_r(t) = \Delta\theta$

- you already know how to solve in the time domain, so I'll solve it by transform.

transform the DE :

$$s^2 \Theta(s) - s \Theta(0^-) - \dot{\Theta}(0^-) + \frac{1}{\tau_0} (s \Theta(s) - \Theta(0^-)) + \frac{A K_0}{\tau_0} \Theta(s) = \frac{A K_0}{\tau_0} \frac{\Delta \theta}{s}$$

$$\left(s^2 + \frac{1}{\tau_0} s + \frac{A K_0}{\tau_0}\right) \Theta(s) = \frac{A K_0 \Delta \theta}{\tau_0 s} + \dot{\Theta}(0^-) + \left(s + \frac{1}{\tau_0}\right) \Theta(0^-)$$

$$\Theta(s) = \frac{\Delta \theta A K_0 / \tau_0}{s(s^2 + \frac{1}{\tau_0} s + \frac{A K_0}{\tau_0})} + \frac{(\Delta \theta - \theta_c)(s + \frac{1}{\tau_0}) + V_{\max} K_0}{s^2 + \frac{1}{\tau_0} s + \frac{A K_0}{\tau_0}}$$

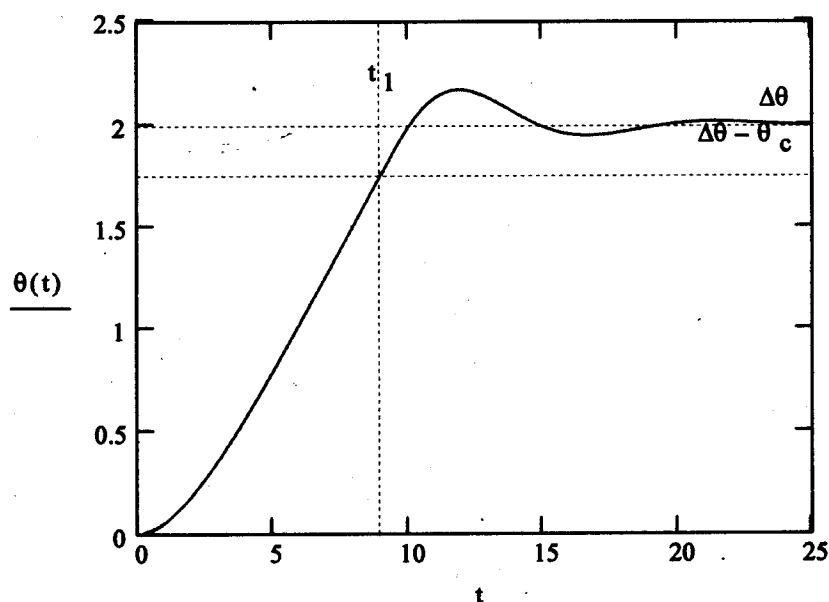
$$= \Delta \theta \left[1 - e^{-t/2\tau_0} \left(\cos(\omega_d t) + \frac{2\omega_d \tau_0}{4A K_0 \tau_0 - 1} \sin(\omega_d t) \right) \right] + e^{-t/2\tau_0} \left[\left(\Delta \theta - \frac{V_{\max}}{A} \right) \cos(\omega_d t) + \frac{2\omega_d \tau_0 \left(\Delta \theta - \frac{V_{\max}}{A} + 2V_{\max} \tau_0 K_0 \right)}{4A K_0 \tau_0 - 1} \sin(\omega_d t) \right]$$

after defining

$$\omega_0 = \sqrt{A K_0 / \tau_0} \quad \zeta = \frac{1}{2\sqrt{A K_0 \tau_0}} \quad \omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

and noting $\theta_c = V_{\max}/A$

$$= \frac{1}{2} \sqrt{\frac{4A K_0}{\tau_0} - \frac{1}{\tau_0^2}}$$



$$K_0 = 0.25$$

$$\tau_0 = 2$$

$$V_{\max} = 1$$

$$A = 4$$

$$\theta_c = 0.25$$

$$\Delta \theta = 2$$

- linear rise up to the switch point
- overshoot is just short of hitting the switch point on the other side.