6.7 Stability

HEW 9:11

- We have seen that feedback modifies the location of the poles — and, if they move into the RHP, the system is unstable. Question: how to detect and avoid instability at design time (instead of when you turn the system on!)

- For the first and second order systems we've looked at so far, it has been simple:
  - don't use positive loop gain
  - for PI, keep $A >$ a negative value.

- For third and higher order systems, the party's over. Determining stability can be difficult if we have several interacting design parameters.
  - If there's only one parameter, such as gain A, we can plot the locus of the roots as A increases and note where it crosses $j\omega$ axis.

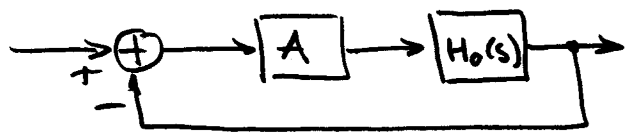
    Finding the roots by computer is useful, but it can't be done by hand and it's not very intuitive.

Many practical systems do have order > 2!

Example of locus of roots. Suppose open loop system is

$$H_o(s) := \frac{1}{(s+1)\cdot(0.5\cdot s+1)\cdot(0.25\cdot s+1)}$$

How large can we make the gain A in the closed loop system?



Closed loop transfer function is $H(s) = \dfrac{A\cdot H(s)}{1+A\cdot H(s)} = \dfrac{A}{((s+1)\cdot(0.5\cdot s+1)\cdot(0.25\cdot s+1)+A)}$

Our concern is the location of the poles of H(s); that is, the roots of the denominator. Mathcad symbolics gives the polynomial coefficients of the denominator

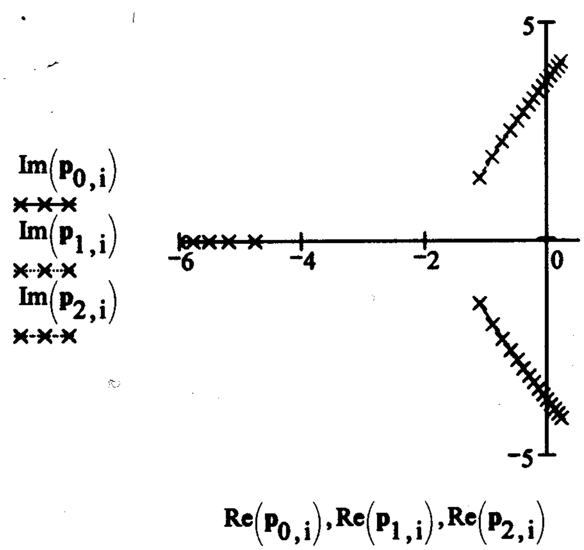
$(s+1)\cdot(0.5\cdot s+1)\cdot(0.25\cdot s+1)+A$ as $v(A) := (1+A \;\; 1.75 \;\; 0.875 \;\; 0.125)^T$

so that pole locations are

$$\text{poles}(A) := \left|\begin{array}{l} p \leftarrow \text{polyroots}(v(A)) \\ p' \leftarrow \text{augment}(p, \text{Im}(p)) \\ p' \leftarrow \text{csort}(p', 1) \\ \text{submatrix}(p', 0, 2, 0, 0) \end{array}\right.$$

this just sorts the roots so they don't swap positions

$i := 0..14$ $\quad A_i := 1\cdot(i+1)$ $\quad p^{\langle i\rangle} := \text{poles}(A_i)$ $\quad k := 0..2$



It looks as though A=10 or so is the limit.

(Actual limit is A=11.25)

- One test for stability is the <u>Routh-Hurwitz criterion</u>

$\boxed{\text{H\&VV 9.12}}$

  - This test tells you how many roots of some $A(s)$ lie in the RHP.
  - If $A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$ then form the first two rows of the Routh array

$$\begin{array}{lllll} \text{row } n: & a_n & a_{n-2} & a_{n-4} & \cdots \\ \text{row } n-1: & a_{n-1} & a_{n-3} & a_{n-5} & \end{array} \left\{ \begin{array}{ccc} a_1 & \text{or} & a_0 \\ a_0 & & 0 \end{array} \right\} \text{ dep on } n \text{ even or odd}$$

Next row is

$$\text{row } n-2: \quad \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \quad \cdots$$

Same pattern for following rows. The number of sign changes is the number of RHP poles.

  - <u>Example</u> 8. poly from example one page back

$$s^3 + 7s^2 + 14s + 8(1+A)$$

for $A=9$:

$$\begin{array}{lll} 3: & 1 & 14 \\ 2: & 7 & 72 \\ 1: & 3.7 & \\ 0: & 72 & \end{array} \qquad \frac{7 \times 14 - 1 \times 72}{7}$$

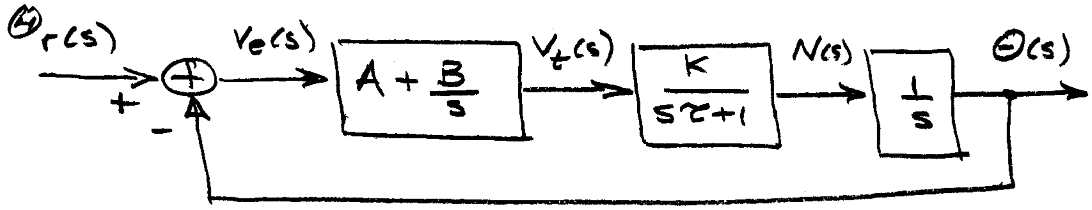
no sign changes
no RHP poles
stable

for $A = 12$

$$\begin{array}{lll} 3: & 1 & 14 \\ 2: & 7 & 104 \\ 1: & -0.857 & \\ 0: & 104 & \end{array} \qquad 8(1+A) \qquad \frac{7 \times 14 - 1 \times 104}{7}$$

two sign changes
two RHP poles
unstable

– Example: Use of PI control in a simple position control system (not a good idea, but it's an example).



$$H(s) = \frac{\Theta(s)}{\Theta_r(s)} = \frac{\dfrac{(As+B)K}{s^2(s\tau+1)}}{1 + \dfrac{(As+B)K}{s^2(s\tau+1)}} = \frac{(As+B)K}{s^3\tau + s^2 + KAs + KB}$$

What constraints on A, B for stability?

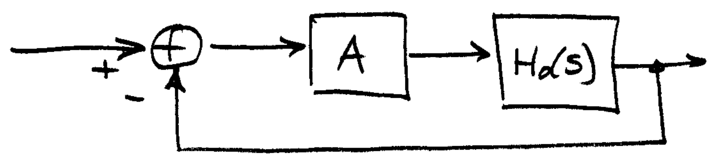
- The Routh Hurwitz test is very useful, but it does not show how close to disaster you are, nor does it give you any insight

- A very useful approximate method that
  - shows you what's going on
  - and suggests what to do about it

  is the Bode plot with <u>gain and phase margins</u>. It is not restricted to polynomial transfer functions (for example, it allows delay blocks).

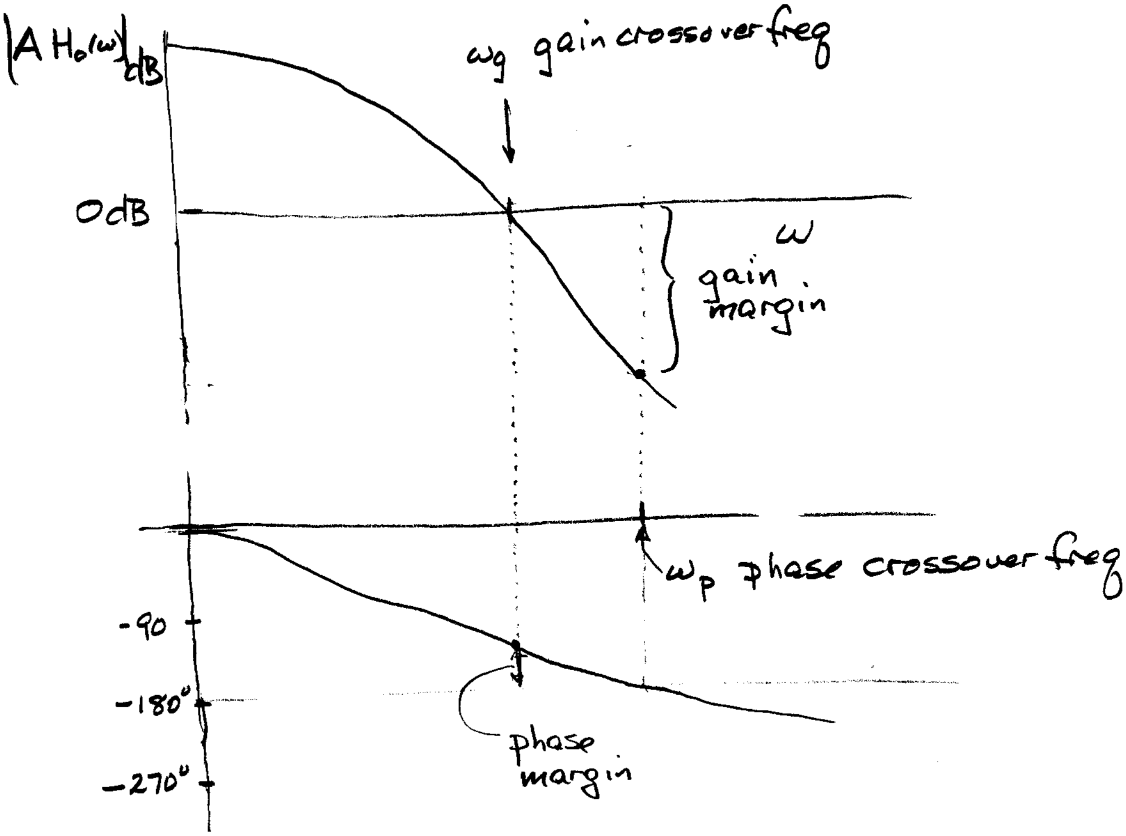
$\boxed{\text{H\&vV 9.16}}$

- Bode plot rationale. Consider the closed loop system:



- Instability is a result of positive feedback: a signal is fed back with $|gain| \geq 1$ and negative sign (reversed in the subtractor). It can recirculate and grow
- Any frequency for which $|A\, H_o(\omega)| \geq 1$ and $\arg[H_o(\omega)] = 180°$ will grow without bound.
- Necessary condition for stability is that no frequency meet that condition

- So Bode plot method is tied to the frequency response of the loop gain $A\,H_o(\omega)$



- Note that the gain crossover freq $\omega_g < \omega_p$ the phase crossover; if it were greater, then the system would be unstable at ω_p and its neighbourhood.
- Both the gain margin and phase margin must be positive. Values of 10dB and 45° are common and conservative choices
- The "order-3 and higher" focus of discussion is now clear (order 2 reaches −180° only at $\omega \to \infty$)
- Gain A_{dB} shifts the magnitude plot up and down. This lets us set a maximum A.

Example of gain and phase margins. Consider the open loop transfer function

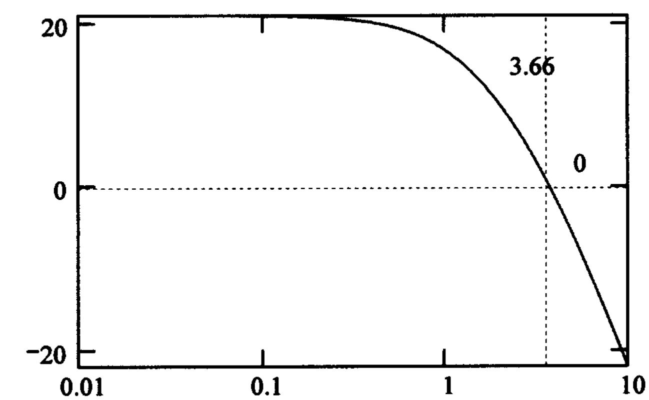
$$H_o(s) := \frac{1}{(s+1)\cdot(0.5\cdot s+1)\cdot(0.25\cdot s+1)} \qquad G(s,A) := A\cdot H_o(s)$$

Plot the open loop frequency response. Asymptotic Bode diagrams (the approximate sketch method) are appropriate for pencil and paper work. I'll do it by computer.

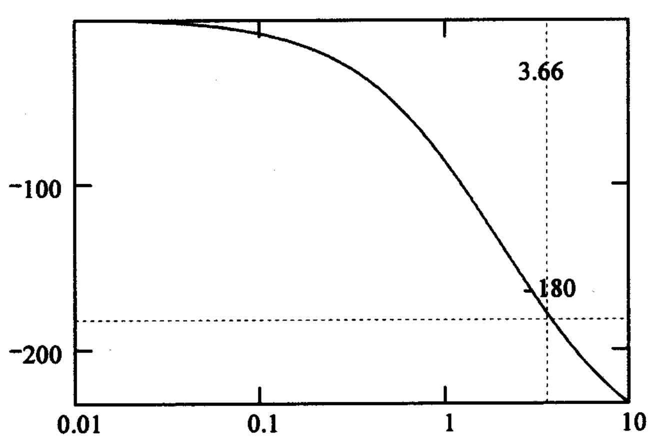
$$dB(x) := \mathrm{if}\left(x>10^{-14}, 10\cdot\log(x), -140\right) \qquad nat(x) := 10^{0.1\cdot x}$$

$$ph(\alpha) := \frac{-180}{\pi}\cdot\mathrm{angle}(\mathrm{Re}(\alpha), -\mathrm{Im}(\alpha)) \qquad \omega := 0.02, 0.04 .. 10$$

The first plot will be with $A := 11.25$ the value we previously determined to be the limit

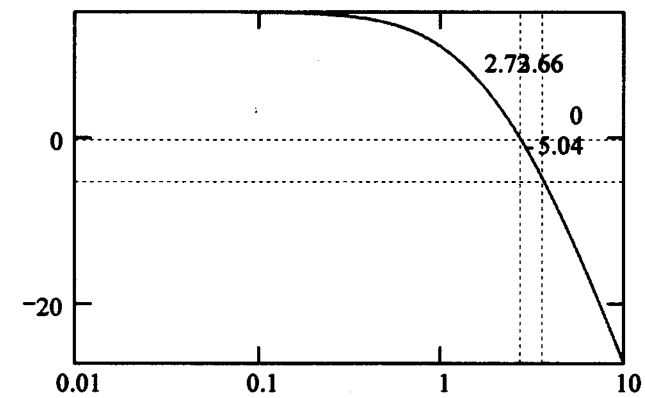


For this choice of A, the gain and phase crossover frequencies are identical, showing that the system is *just* unstable.

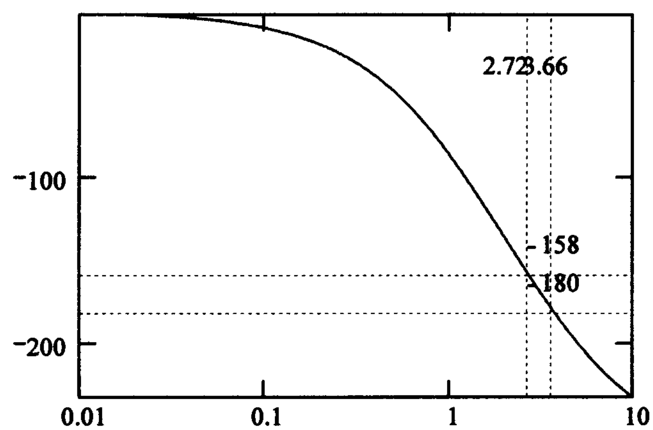


6.7.8

Next try a more cautious design, with A := 6



gain margin 5 dB



phase margin 22 degrees

Since the choice of A simply shifts the magnitude plot up and down, it is relatively simple to calculate how the gain and phase margins depend on A.