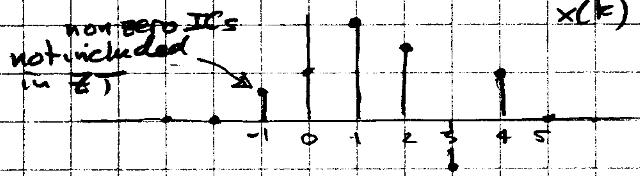
7.3 Some Properties of Z Transform

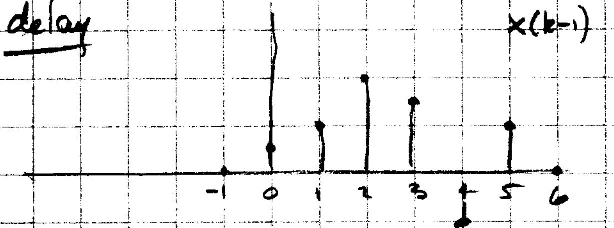
$\boxed{\text{H\&vV 7.4}}$

- Time shift properties



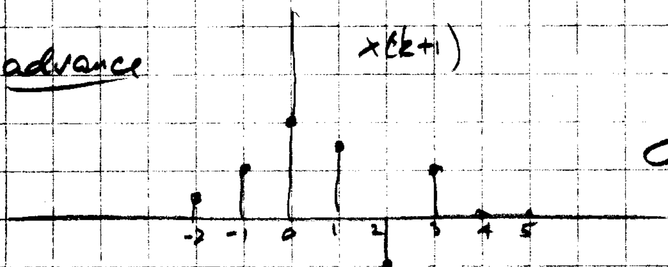
$$\mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k) z^{-k}$$

delay



$$\mathcal{Z}[x(k-1)] = \sum_{k=0}^{\infty} x(k-1) z^{-k}$$
$$= \sum_{i=-1}^{\infty} x(i) z^{-(i+1)}$$
$$= z^{-1} X(z) + x(-1)$$

advance



$$\mathcal{Z}[x(k+1)] = \sum_{k=0}^{\infty} x(k+1) z^{-k}$$
$$= \sum_{i=1}^{\infty} x(i) z^{-(i-1)} = z \sum_{i=1}^{\infty} x(i) z^{-i}$$
$$= z\left(X(z) - x(0)\right) = z X(z) - z\,x(0)$$

- Multiplication - Convolution. z transforms are polynomials in z^{-1}, with coefficients equal to the time samples. What happens if you multiply two z-transforms? $Y(z) = X(z)H(z)$

$$
\begin{array}{llllll}
 & x(0) & + x(1) z^{-1} & + x(2) z^{-2} & + x(3) z^{-3} & \\
 & h(0) & + h(1) z^{-1} & + h(2) z^{-2} & + h(3) z^{-3} & \\
\hline
 & h(0)x(0) & + h(0)x(1) z^{-1} & + h(0)x(2) z^{-2} & + h(0)x(3) z^{-3} & + \\
 & & + h(1)x(0) z^{-1} & + h(1)x(1) z^{-2} & + h(1)x(2) z^{-3} & + \\
 & & & + h(2)x(0) z^{-2} & + h(2)x(1) z^{-3} & + \\
 & & & & + h(3)x(0) z^{-3} & + \cdots \\
\hline
 & y(0) & + y(1) z^{-1} & + y(2) z^{-2} & + y(3) z^{-3} & + \cdots
\end{array}
$$

The y(k) samples follow a convolution pattern

$$y(k) = \sum_{i=0}^{k} h(i)\, x(k-i)$$

Hence

$$y(k) = h(k) \circledast x(k) \iff Y(z) = H(z)X(z).$$

We can show this same result more formally by

$$y(k) = \sum_{i=0}^{\infty} h(i)\, x(k-i) \qquad (\text{after defining } x(n)=0,\ n<0)$$

$$Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k} = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} h(i)\,x(k-i)\,z^{-k} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} h(i)x(k-i)\,z^{-k}$$

$$= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} h(i)\,x(l)\, z^{-(l+i)} \qquad (l = k-i,\ \text{recall } x(l)=0 \text{ for } l<0)$$

$$= \sum_{i=0}^{\infty} h(i)\, z^{-i}\, X(z) = H(z)\, X(z)$$

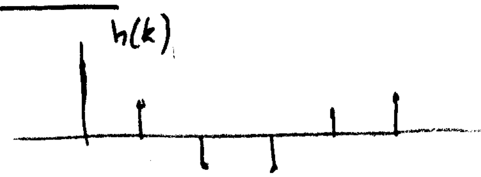
$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

H(z) is the transfer function.
Not necessarily a rational polynomial

$\boxed{\text{H \& VV 7.6}}$

7.4 Frequency Response

- Discrete-time filter



If impulse-weighted in continuous time then

$$h_c(t) = \sum_{k=0}^{\infty} h(k)\,\delta(t-kT), \qquad H(s) = \sum_k h(k)\, e^{-skT} = \sum_k h(k)\, z^{-k}$$

Therefore the frequency response

$$H(\omega) = \mathcal{L}\left[h_c(t)\right]_{s=j\omega} = \mathcal{Z}\left[h(k)\right]_{z=e^{j\omega T}}$$

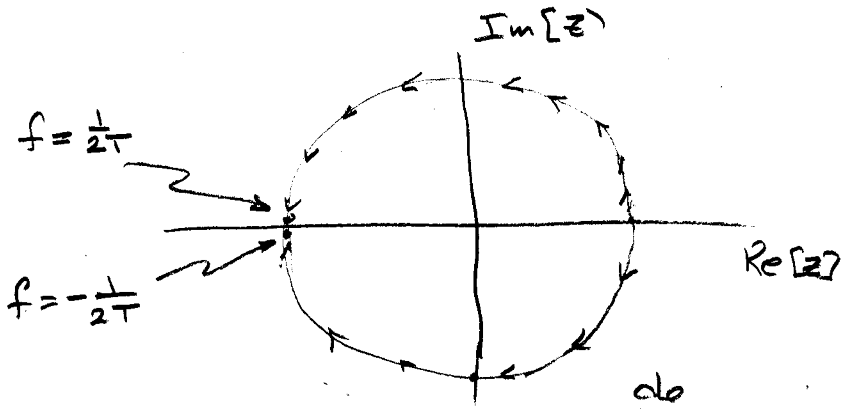
or $z = e^{j2\pi f T}$

- For discrete time systems

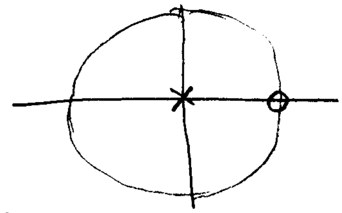
$$
\begin{tikzcd}
 & h(t) = \sum_{k=0}^{\infty} x(k)\, \delta(t-kT) \arrow[dl, "\mathcal{F}"'] \arrow[dr, "\mathcal{L}"] & \\
H(f) = \sum_{k=0}^{\infty} x(k)\, e^{-j2\pi f kT} \arrow[rr, leftrightarrow, "s=j2\pi f"] \arrow[dr, leftrightarrow, "z=e^{j2\pi fT}"'] & & H(s) = \sum_{k=0}^{\infty} x(k)\, e^{-skT} \arrow[dl, leftrightarrow, "z=e^{sT}"] \\
 & H(z) = \sum_{k=0}^{\infty} x(k)\, z^{-k} &
\end{tikzcd}
$$

From study of sampling, we expect frequency response to be periodic with period $1/T$. Is it?

$H(e^{j2\pi fT})$ is periodic since $e^{j2\pi fT}$ is.



Example: $h(k) = (1, -1, 0, 0, ..)$ with $f_s := 1$ and $t_s := \dfrac{1}{f_s}$



The z-transform and Fourier transform (i.e., frequency response) are

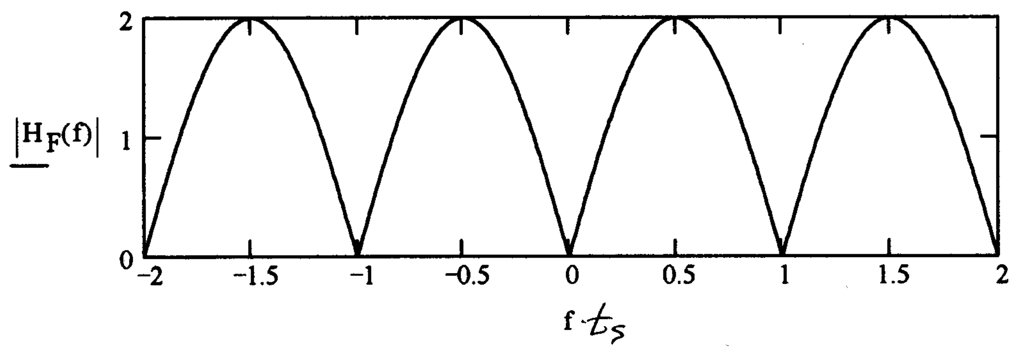
$$H_Z(z) := 1 - z^{-1} \qquad H_F(f) := 1 - e^{-j \cdot 2\cdot\pi\cdot f\cdot t_s} \qquad \text{since } z = e^{j \cdot 2\cdot\pi\cdot f\cdot t_s}$$

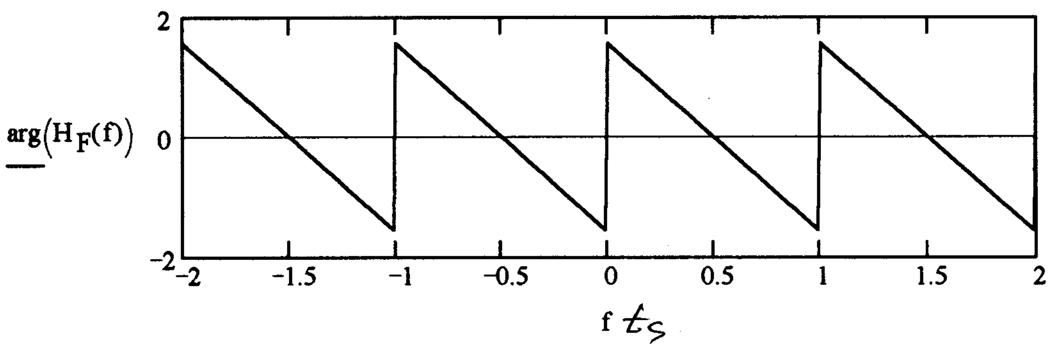
The magnitude and phase are

$$\left|H_F(f)\right| = \sqrt{H_F(f)\cdot\overline{H_F(f)}} = \sqrt{2 - 2\cdot\cos\left(2\cdot\pi\cdot f\cdot t_s\right)} = \sqrt{4\cdot\sin\left(\pi\cdot f\cdot t_s\right)^2} = 2\cdot\left|\sin\left(\pi\cdot f\cdot t_s\right)\right|$$

$$\arg\left(H_F(f)\right) = \operatorname{atan}\left(\frac{\sin\left(2\cdot\pi\cdot f\cdot t_s\right)}{1 - \cos\left(2\cdot\pi\cdot f\cdot t_s\right)}\right)$$

$$f := -2\cdot f_s, -1.99\cdot f_s .. 2\cdot f_s$$



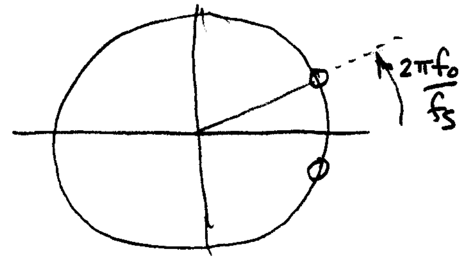


Notes: (1) It removes DC; (2) Frequency response is periodic - no surprise, since equispaced discrete in one domain implies periodic in the other.

Example: Design an FIR filter with a null at a specified frequency $f_o := 0.2$ normalized by the sampling frequency - that is, a null at 0.2 f_s. The sampling frequency is also normalized with $f_s := 1$

We want zeroes on the unit circle at that sinusoid frequency. From Section 7.2 (using zeroes instead of poles), we construct H(z) as

$$\left(z - e^{j\cdot 2\cdot\pi\cdot\frac{f_o}{f_s}}\right)\cdot\left(z - e^{-j\cdot 2\cdot\pi\cdot\frac{f_o}{f_s}}\right) = z^2 - 2\cdot\cos\left(2\cdot\pi\cdot\frac{f_o}{f_s}\right)\cdot z + 1$$

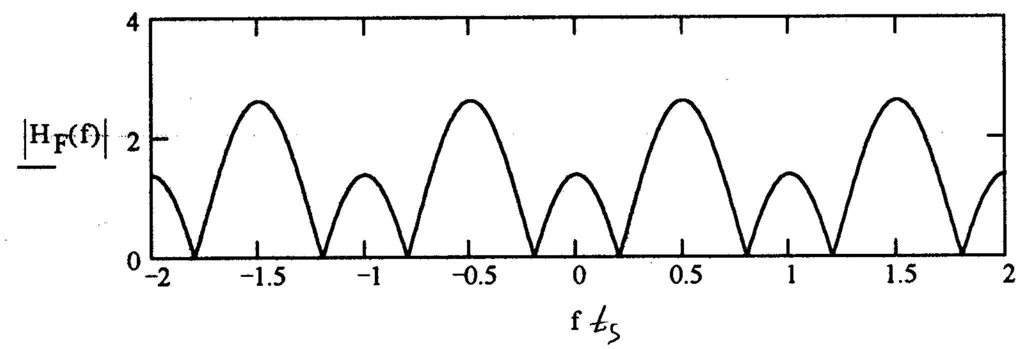


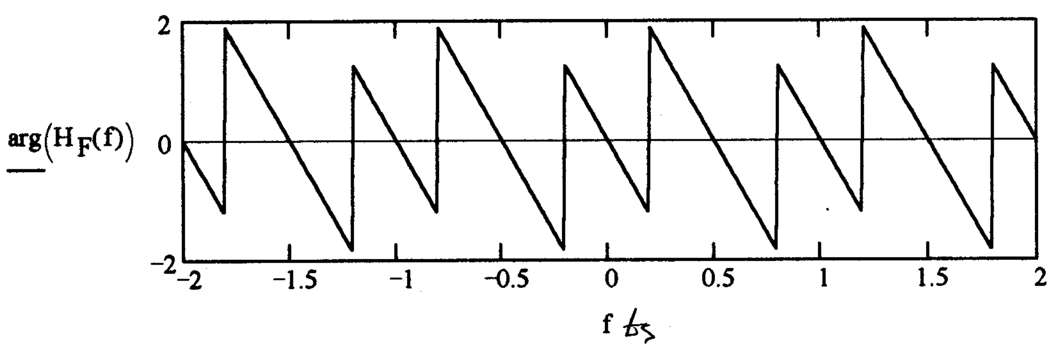
This filter is not causal (z^2 and z terms mean its impulse response is nonzero for negative time) so delay it by 2 steps: multiply by z^{-2}. Then the z-transform, impulse response and Fourier transform (i.e., frequency response) are

$$H_Z(z) := 1 - 2\cdot\cos\left(2\cdot\pi\cdot\frac{f_o}{f_s}\right)\cdot z^{-1} + z^{-2} \qquad h := \begin{bmatrix} 1 \\ -2\cdot\cos\left(2\cdot\pi\cdot\frac{f_o}{f_s}\right) \\ 1 \end{bmatrix}$$

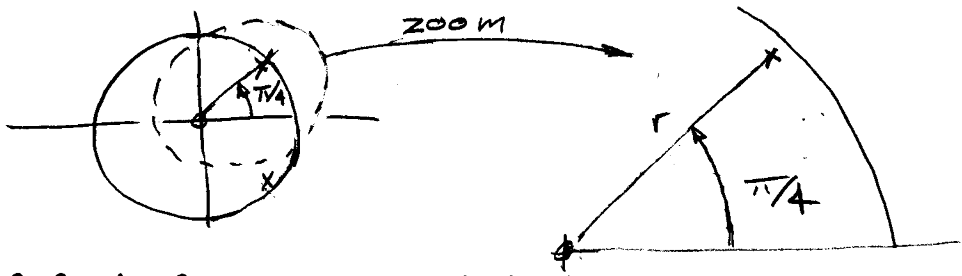
$$H_F(f) := H_Z\left(e^{j\cdot 2\cdot\pi\cdot\frac{f}{f_s}}\right) \qquad \text{since } z = e^{j\cdot 2\cdot\pi\cdot f\cdot t_s}$$

$$f := -2\cdot f_s, -1.99\cdot f_s .. 2\cdot f_s$$





Example: Examine a 2nd order IIR digital filter that resonates at 1/8 of the sampling frequency $f_s := 1$.
As shown in Section 7.2, the poles are close to the unit circle with an angle of $2\pi f_0/f_s = \pi/4$, and the radius r controls the Q. The pole-zero diagram is shown below

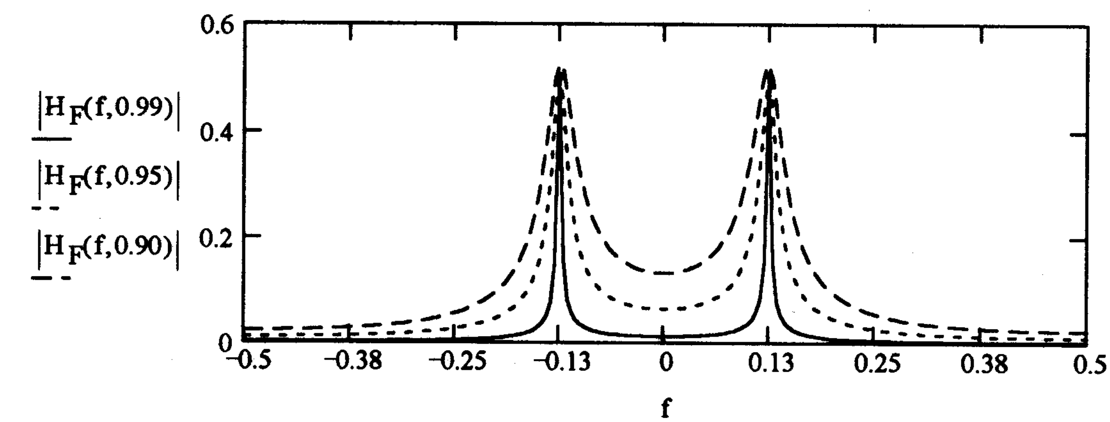


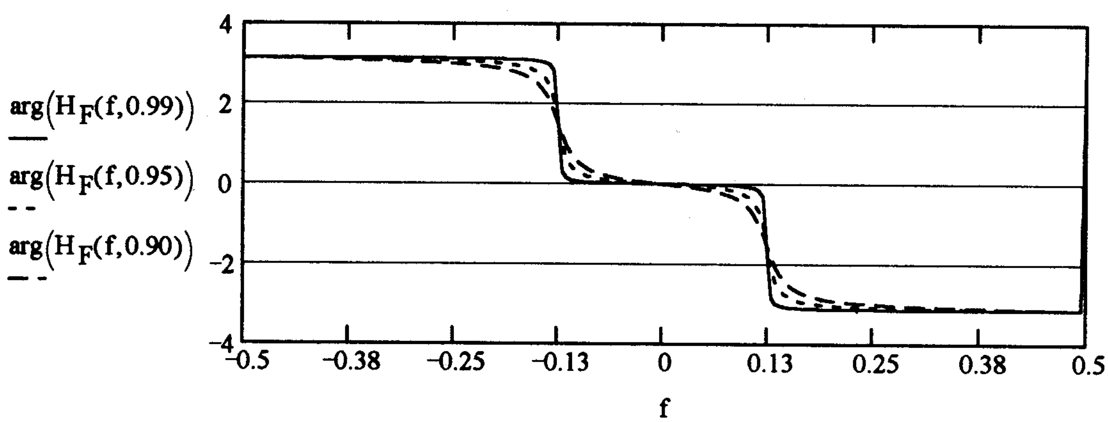
The transfer function, frequency response and unit pulse response, all parameterized by r, are

$$H_Z(z,r) := \frac{1}{\sqrt{2}}\cdot\frac{(1-r)\cdot z}{z^2 - z\cdot\sqrt{2}\cdot r + r^2} \qquad H_F(f,r) := H_Z\left(e^{j\cdot 2\cdot\pi\cdot\frac{f}{f_s}}, r\right) \qquad h(k,r) := (1-r)\cdot r^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)$$

The unit pulse response comes from inversion of the transform (actually, I went the other way, starting with h(k) and getting H(z) from it)

$$f := -\frac{f_s}{2}, -0.99\cdot\frac{f_s}{2} .. \frac{f_s}{2}$$ expanded frequency scale compared with previous examples





7.5 More Properties

Mult by k

$$\mathcal{Z}_I\left(k\,x(k)\right) = \sum_{k=0}^{\infty} k\,x(k)\,z^{-k}$$

$$= -z \sum_{k=0}^{\infty} k\,x(k)\,z^{-k-1}$$

$$= -z\,\frac{d}{dz}X(z)$$

Initial Value

$$x(0) = \lim_{z\to\infty} X(z)$$

Final Value

if it exists $\displaystyle\lim_{N\to\infty} x(N) = \lim_{z\to 1} (z-1)\,X(z)$

Pf $\quad \mathcal{Z}_I\left[x(k+1) - x(k)\right] = z\,X(z) - z\,x(0) - X(z)$

$$= (z-1)\,X(z) - z\,x(0)$$

$$= \sum_{k=0}^{\infty}\left(x(k+1) - x(k)\right) z^{-k}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N-1}\left(x(k+1) - x(k)\right) z^{-k}$$

taking $\lim_{z\to 1}$

$$\lim_{N\to\infty}\left(x(N) - x(0)\right) = \lim_{z\to 1} (z-1)\,X(z) - z\,x(0)$$

so $\displaystyle\lim_{N\to\infty} x(N) = \lim_{z\to 1} (z-1)\,X(z)$