7.6 Inversion of Z-Transforms

- Given $X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$ as a function of z.

How can we recover the time sequence $x(0), x(1) \dots$?

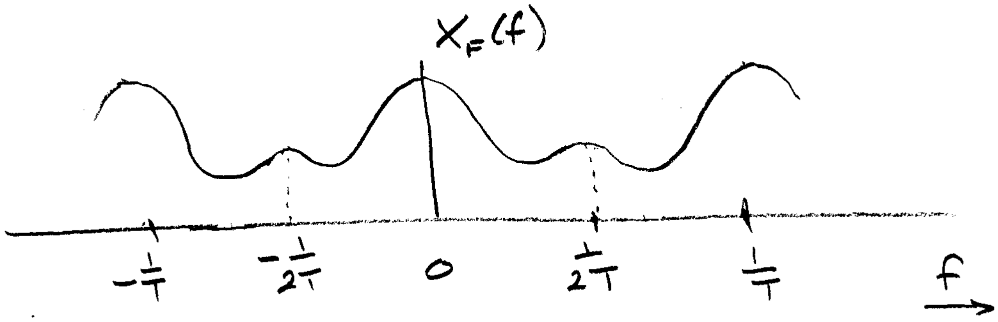
There are several methods, each with advantages and drawbacks.

Fourier Series Method

- This one is indirect, but quite general. To start, assume that the unit circle is in the region of convergence. Then the Fourier transform $X_F(f) = \mathcal{F}\left[\sum_k x(k)\delta(t-kT)\right]$ is obtained by substituting $z = e^{j2\pi fT}$ in $X_z(z)$. So

$$X_F(f) = X_z(e^{j2\pi fT}).$$

It's periodic in f

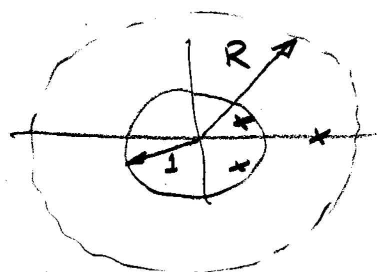


- So expand the frequency response in a Fourier series

$$X_F(f) = \sum_{k=0}^{\infty} x(k)\, e^{-j2\pi f kT} \longleftrightarrow x(t) = \sum_{k=0}^{\infty} x(k)\,\delta(t-kT)$$

$$x(k) = T\int_{-\frac{1}{2T}}^{1/2T} X_F(f)\, e^{j2\pi f kT}\, df \;,\quad k \geq 0$$

- Generalizations:
  - It allows non-causal functions ($x(k) \neq 0,\ k<0$) is unit circle is in ROC
  - It allows poles outside the unit circle if we put integration circle outside poles (and therefore in ROC)

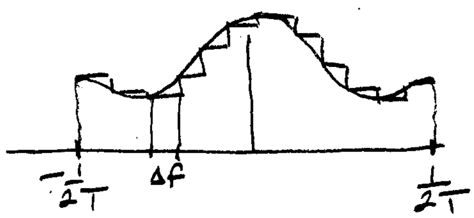


$$z = R e^{j2\pi f T}$$

$$x(k) = T R^k \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X_z(Re^{j2\pi fT})\, e^{j2\pi f kT}\, df$$

- It can be converted to DFT and FFT
  Approximate the integral with an N-point sum

$$\Delta f = \frac{1}{N}\cdot\frac{1}{T} = \frac{1}{NT}$$



$$x(k) = T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(f)\, e^{j2\pi f kT}\, df \approx T\sum_{i=0}^{N-1} X(i\Delta f)\, e^{j2\pi(i\Delta f)kT}\,\Delta f$$

or $x(k) = \frac{1}{N} \sum_{i=0}^{N-1} X\left(\frac{i}{NT}\right) e^{j 2\pi \frac{ik}{N}}$ the Discrete Fourier Transf.

It can be performed, with low computaternal load, by the FFT if N is highly composite.

- Power Series method

substitute $z = \frac{1}{D}$, so we have

$$X(D) = \sum_{k=0}^{\infty} x(k) D^k$$

and expand as power series about $D=0$;

$$X(D) = X(0) + \left.\frac{dX}{dD}\right|_{D=0} D + \frac{1}{2!} \left.\frac{d^2X}{dD^2}\right|_{D=0} D^2 + \cdots$$

Comparison of terms shows

$$x(k) = \frac{1}{k!} \left.\frac{d^k X}{dD^k}\right|_{D=0}$$

Gets messy with all the differentiation. Good for first few, at least. But doesn't need rat poly $X(z)$.

- Special case: $X(D)\Big|_{D=0} = x(0) = X(z)\Big|_{z\to\infty}$

Init Value Theorem

ex $X(z) = e^{-\lambda(1-z^{-1})}$

$X(D) = e^{-\lambda(1-D)} = e^{-\lambda} e^{\lambda D}$

$x(0) = e^{-\lambda}$ $x(1) = \left.\frac{dX}{dD}\right|_{D=0} = e^{-\lambda}\lambda$

$x(2) = \frac{1}{2!} e^{-\lambda} \lambda^2$ etc.

• Long Division Method

If $X(z)$ is a rational polynomial, then divide numerator by denominator.

example $$X(z) = \frac{3z^2 + z}{z^2 + 3z + 9}$$

first write as $$X(z) = \frac{3 + z^{-1}}{1 + 3z^{-1} + 9z^{-2}}$$

then

$$
\begin{array}{r|llll}
 & 3 & -8z^{-1} & -3z^{-2} & \text{etc} \\
\hline
1 + 3z^{-1} + 9z^{-2} & 3 & +z^{-1} & & \\
 & 3 & +9z^{-1} & +27z^{-2} & \\
\cline{2-4}
 & & -8z^{-1} & -27z^{-2} & \\
 & & -8z^{-1} & -24z^{-2} & -72z^{-3} \\
\cline{3-5}
 & & & -3z^{-2} & -72z^{-3} \\
 & & & -3z^{-2} & -9z^{-3} \quad -27z^{-2} \\
\cline{4-5}
 & & & & \text{etc}
\end{array}
$$

advantage – gives first few easily
– no factoring of denom

disadvantage – no general form for $x(k)$

• Residue Method

– This method gives general form, but requires factoring to find poles

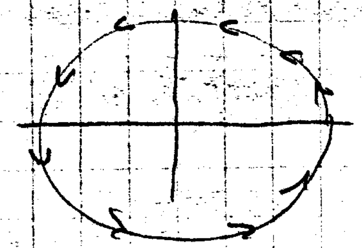
$$x(k) = \sum_i Res_i \left[X(z) z^{k-1} \right]$$

Rationale: $$X(z) = \sum_{i=0}^{\infty} x(i) z^{-i}$$ converges for $|z|$ large enough

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

$$X(z) z^{k-1} = x(0) z^{k-1} + \cdots + x(k-1) + \frac{x(k)}{z} + \frac{x(k+1)}{z^2} + \cdots$$

from complex variables

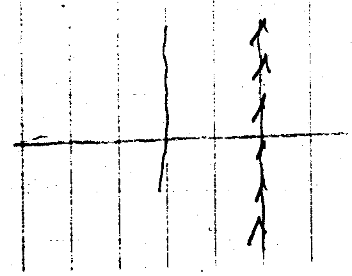


$$\oint X(z) z^{k-1} dz$$

$$= 0 + 0 + \cdots + 2\pi j\, x(k) + 0 + \cdots\cdots = 2\pi j\, x(k)$$

$$x(k) = \frac{1}{2\pi j} \oint X(z) z^{k-1} dz = \sum_{\substack{\text{poles in} \\ \text{the circle}}} Res_i \left[X(z) z^{k-1} \right]$$

can be related to Laplace inversion integral:



since vertical line to right of all poles is equiv to circle enclosing all poles.

<u>example</u> $X(z) = \dfrac{z^2}{(z+1)(z-2)}$ two simple poles (see Appendix)

$$x(k) = \sum_i Res_i \left[\frac{z^{k-1} z^2}{(z+1)(z-2)} \right] = \sum_i Res_i \left[\frac{z^{k+1}}{(z+1)(z-2)} \right]$$

- at $z=-1$ $\quad Res = \left(\dfrac{z^{k+1}}{z-2} \right)\Big|_{z=-1} = \dfrac{(-1)^{k+1}}{-3} = \dfrac{1}{3}(-1)^k \quad k \geq 0$

- at $z = 2$ $\quad Res = \left(\dfrac{z^{k+1}}{z+1} \right)\Big|_{z=2} = \dfrac{2^{k+1}}{3} = \dfrac{2}{3} 2^k \quad k \geq 0$

so $x(k) = \dfrac{1}{3}(-1)^k + \dfrac{2}{3} 2^k, \quad k \geq 0$

Check with IVT:

example $X(z) = \dfrac{z+1}{z^2-6z+25}$ poles at $3 \pm j4$ z_0, z_0^*

$= 5e^{\pm j\,0.93}$

$$x(k) = \sum_i \mathrm{Res}_i \left[\frac{z^k (z+1)}{z(z-z_0)(z-z_0^*)} \right]$$

Res at 0 : $\frac{1}{25}\,\delta(k)$

$$\mathrm{Res}_{z_0} + \mathrm{Res}_{z_0^*} = \mathrm{Res}_{z_0} + \mathrm{Res}^*_{z_0} = 2\,\mathrm{Re}\left[\mathrm{Res}_{z_0}\right]$$

$$= 2\,\mathrm{Re}\left[\frac{z_0^k\,(z_0+1)}{z_0\,(z_0 - z_0^*)} \right]$$

$$= 2\,\mathrm{Re}\left[\frac{5^k e^{jk0.93}(4+j4)}{5e^{j0.93}(j8)} \right]$$

$$= \frac{5^{k-1}}{4}\,\mathrm{Re}\left[-j\,e^{j(k-1)0.93}(4+j4) \right]$$

$$= \sqrt{2}\,5^{k-1}\cos\left(0.93(k-1) - \frac{\pi}{4}\right)$$

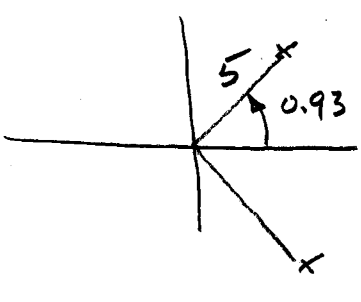
$$= 0.28\;5^k \cos(0.93k - 1.7)$$

total :

$$x(k) = 0.04\,\delta(k) + 0.28\;5^k\cos(0.93k - 1.7)$$

check $x(0) = 0$ as required by IVT.

Again radius of complex poles determines the decay (growth) rate r^k



and the angle determines the freq.

$$e^{jk\theta} = \cos(k\theta) + j\sin(k\theta)$$

example a double pole

$$X(z) = \frac{z^2}{(z+1)^2(z-2)}$$

$$x(k) = \sum \mathrm{Res}_i \left[\frac{z^{k-1} z^2}{(z+1)^2(z-2)} \right] = \sum \mathrm{Res}_i \left[\frac{z^{k+1}}{(z+1)^2(z-2)} \right]$$

- at $z=-1$:

$$\frac{d}{dz} \frac{z^{k+1}}{z-2} = \frac{(k+1) z^k (z-2) - z^{k+1}}{(z-2)^2}$$

$$\mathrm{Res} = \left. \frac{d}{dz} \frac{z^{k+1}}{z-2} \right|_{z=-1} = \frac{(k+1)(-1)^k(-3) - (-1)^{k+1}}{(-3)^2} \qquad k \geq 0$$

$$= \frac{-3(k+1)(-1)^k + (-1)^k}{9} = \frac{(-3k-2)(-1)^k}{9} \qquad k \geq 0$$

- at $z=2$

$$\mathrm{Res} = \left. \frac{z^{k+1}}{(z+1)^2} \right|_{z=2} = \frac{2^{k+1}}{3^2} = \frac{2}{9} 2^k \qquad k \geq 0$$

so

$$x(k) = \frac{-3k-2}{9}(-1)^k + \frac{2}{9} 2^k$$

example $X(z) = \frac{1}{(z+1)(z-2)}$ (gives $\delta(k)$)

$$x(k) = \sum_i \mathrm{Res}_i \left[\frac{z^{k-1}}{(z+1)(z-2)}\right] = \sum \mathrm{Res}\left[\frac{z^k}{z(z+1)(z-2)}\right]$$

- poles at $-1, 2$ are easy:

$$\frac{(-1)^k}{(-1)(-3)} + \frac{2^k}{2 \cdot 3}$$

- but at 0, we have $\frac{0^k}{(1)(-2)}$

this is 0 for $k>0$. What about $k=0$?

We'll take $\lim_{z\to 0} \frac{z^0}{(z+1)(z-2)} = -\frac{1}{2}$.

Residue $= -\frac{1}{2}\delta(k)$

$$x(k) = \frac{1}{3}(-1)^k + \frac{1}{6} 2^k - \frac{1}{2}\delta(k), \quad k \geq 0$$

k	x
0	0
1	0
2	1
⋮	⋮

example same as above, but avoid the pole at zero by noting that

$$\sum \mathrm{Res}\left[\frac{z^k z^{-1}}{(z+1)(z-2)}\right]$$

is a one step delay of $\sum \mathrm{Res}\left[\frac{z^k}{(z+1)(z-2)}\right]$, which we calculate as:

$$-\frac{1}{3}(-1)^k + \frac{1}{3} 2^k$$

Delay it: $x(k) = -\frac{1}{3}(-1)^{k-1} + \frac{1}{3} 2^{k-1} \qquad k \geq 1$

$$= \frac{1}{3}(-1)^k + \frac{1}{6} 2^k$$

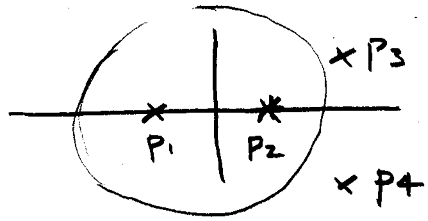
and $x(k) = 0, \quad k = 0$

k	x
0	0
1	0
2	1

So we get same result.

Appendix How to calculate residues.

Suppose $F(z) = \dfrac{n(z)}{d(z)}$ has poles



p_1 simple

p_2 double

p_3, p_4 conjugate pairs

Find all the residues.

① Factor the denominator

$$F(z) = \frac{n(z)}{(z-p_1)(z-p_2)^2(z-p_3)(z-p_3^*)}$$

② For a simple pole, remove the pole

$$G_{p_1}(z) = (z-p_1)F(z) \quad \text{or} \quad F(z) = \frac{G_{p_1}(z)}{(z-p_1)}$$

and evaluate the rest at the pole

$$\text{Res}_{p_1}[F(z)] = G_{p_1}(z)\Big|_{z=p_1} = G_{p_1}(p_1) = \frac{n(p_1)}{(p_1-p_2)^2(p_1-p_3)(p_1-p_3^*)}$$

③ Do the same for a simple complex pole

$$\text{Res}_{p_3}[F(z)] = G_{p_3}(z)\Big|_{z=p_3} = G_{p_3}(p_3) = \frac{n(p_3)}{(p_3-p_1)(p_3-p_2)^2(p_3-p_3^*)}$$

Note $\text{Res}_{p_3^*}[F(z)] = \text{Res}_{p_3}[F(z)]^*$

so $\text{Res}_{p_3}[F(z)] + \text{Res}_{p_3^*}[F(z)] = 2\,\text{Re}\left[\text{Res}_{p_3}[F(z)]\right]$

(4) For a double pole, remove the pole, and differentiate before evaluating.

$$G_2(z) = \frac{n(z)}{(z-p_1)(z-p_3)(z-p_3^*)}$$

$$\underset{p_2}{\text{Res}}\left[F(z)\right] = \left.\frac{d}{dz} G_2(z)\right|_{z=p_2} \qquad \text{(often messy)}$$

For higher order poles, eg. $\dfrac{n(z)}{(z-p_2)^n \cdots}$

$$\text{Res}_{p_2}\left[F(z)\right] = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} G_2(z).$$