7.7 Solving Difference Equations By Z Transform

- Consider the 2^{nd} order ΔE

$$y(k) - 3y(k-1) + 2y(k-2) = u(k) + u(k-2), \quad k \geq 0$$

with ICs $\quad y(-1) = 7 \quad y(-2) = 5$

- As with DEs and Laplace, we can transform it term by term. Recall

$$\mathcal{Z}_I[y(k-1)] = z^{-1}Y(z) + y(-1)$$

$$\mathcal{Z}_I[y(k-2)] = z^{-1}(z^{-1}Y(z) + y(-1)) + y(-2) = z^{-2}Y(z) + z^{-1}y(-1) + y(-2)$$

so the ΔE becomes

$$Y(z) - 3(z^{-1}Y(z) + 7) + 2(z^{-2}Y(z) + 7z^{-1} + 5) = U(z) + z^{-2}U(z)$$

Collect terms:

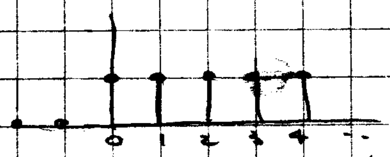
$$Y(z) = \frac{1+z^{-2}}{1-3z^{-1}+2z^{-2}}U(z) + \frac{11-14z^{-1}}{1-3z^{-1}+2z^{-2}}$$

Note z^{-1} vs z

$$= \underbrace{\underbrace{\frac{z^2+1}{z^2-3z+2}}_{H(z)} U(z)}_{zsr} + \underbrace{\frac{(11z-14)z}{z^2-3z+2}}_{zir}$$

- It's even easier with zero ICs, of course. Just read off the coefficients.
- Try a few things with this example
  - initial value of zir: $\lim_{|z|\to\infty} \frac{(11z-14)z}{z^2-3z+2} = 11$

– find zsr if input is a step.



$$U(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$Y_{zs}(z) = \frac{(z^2+1)z}{(z-1)^2(z-2)}$$

$$\sum_i \mathrm{Res}_i \left[Y_{zs}(z)\, z^{k-1} \right]_{p_i}$$

At $z=2$: $\mathrm{Res}_2 = \left.\dfrac{(z^2+1)z^k}{(z-1)^2}\right|_{z=2} = 5 \cdot 2^k \quad k \geq 0$

At $z=1$: $\mathrm{Res}_1 = \left.\dfrac{d}{dz}\dfrac{(z^2+1)z^k}{z-2}\right|_{z=1} = \left.\dfrac{(2z\,z^k + k(z^2+1)z^{k-1})(z-2) - (z^2+1)z^k}{(z-2)^2}\right|_{z=1}$

$= -2k - 4 \quad k \geq 0$

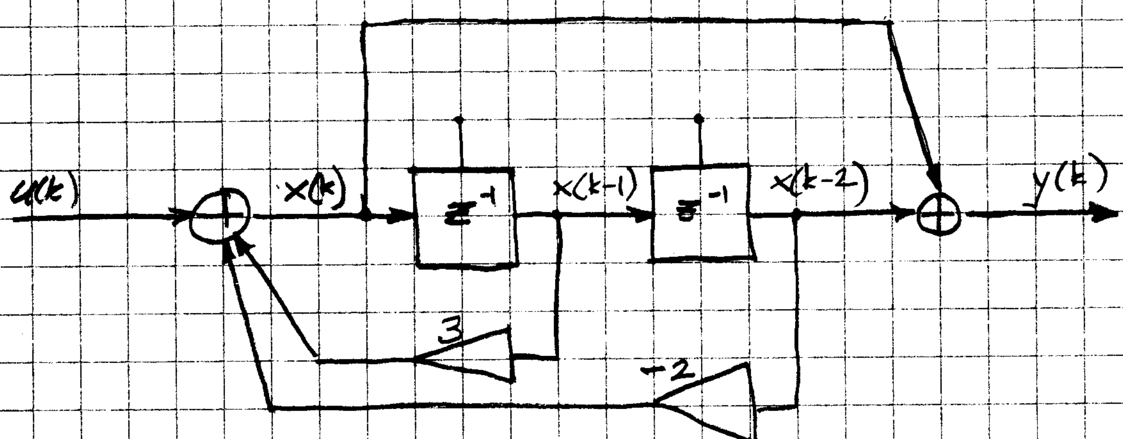
$$y(k) = 5 \cdot 2^k - 2k - 4 \quad k \geq 0$$

check: $y(0) = 1$, IVT says $y(0) = 1$

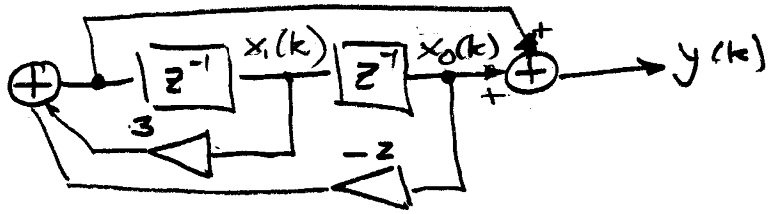
• Now link the ΔE with simulation diagram.

To start, assume IC's are zero. then

$$y(k) = 3y(k-1) - 2y(k-2) + u(k) + u(k-2)$$



- Initial conditions — the initial values held on the delay blocks — require some thought. It's a z.i.r., so redraw the diagram



$$\underline{x}(k) = \begin{bmatrix} x_0(k) \\ x_1(k) \end{bmatrix}$$

Question: what value of $\underline{x}(-1)$ produces the same zir as the IC's on $y(k)$?

$$\underline{x}(k+1) = A\ \underline{x}(k) \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \qquad \text{so } \underline{x}(-2) = A^{-1}\underline{x}(-1)$$

$$A^{-1} = \begin{bmatrix} 3/2 & -\tfrac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$y(k) = C\ \underline{x}(k) \qquad C = \begin{bmatrix} -1 & 3 \end{bmatrix}$$

Match IC's:

$$y(-2) = C\ \underline{x}(-2) = C A^{-1} \underline{x}(-1) = \begin{bmatrix} \tfrac{3}{2} & \tfrac{1}{2} \end{bmatrix} \underline{x}(-1)$$

$$y(-1) = C\ \underline{x}(-1) = \begin{bmatrix} -1 & 3 \end{bmatrix} \underline{x}(-1)$$

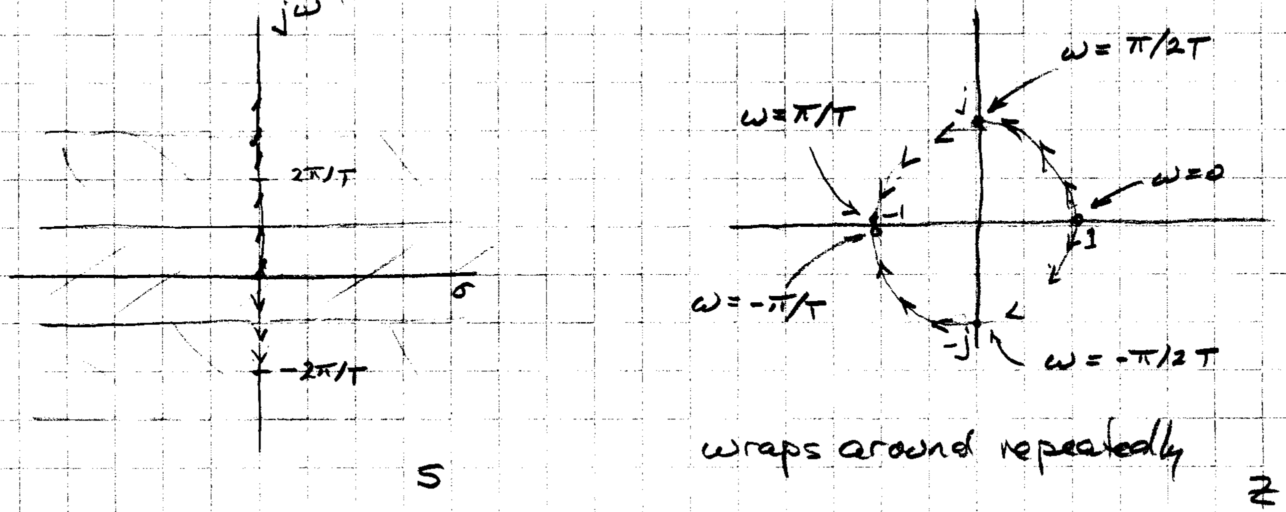
$$\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ -1 & 3 \end{bmatrix} \underline{x}(-1)$$

$$\underline{x}(-1) = \begin{bmatrix} 3/2 & \tfrac{1}{2} \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\ 7/8 \\ 1\ 3/8 \end{bmatrix}$$

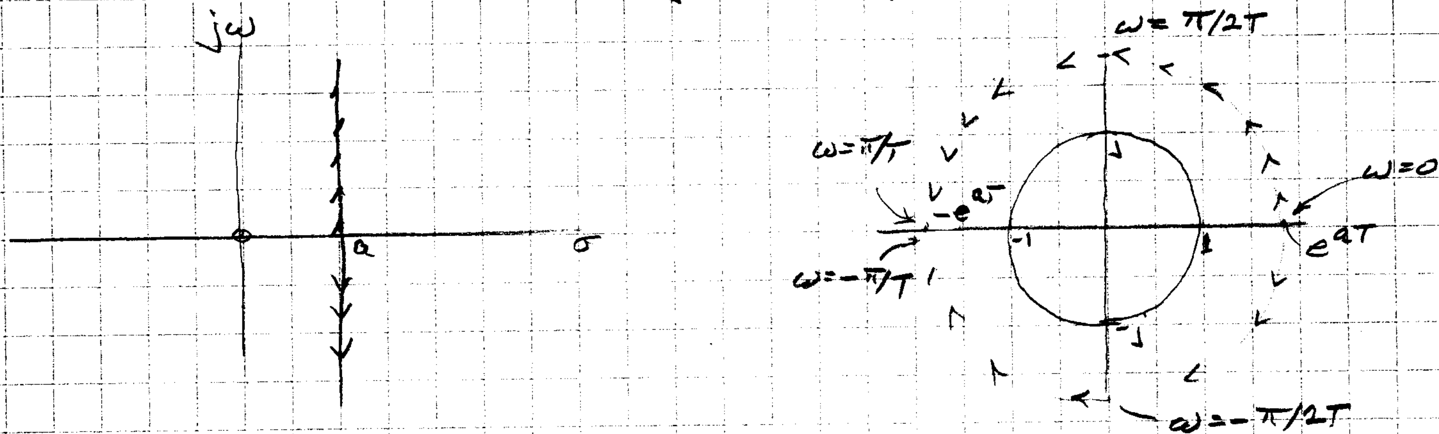
7.8 Links Between Laplace s and Z z

- What is the relation between s and $z = e^{sT} = e^{\sigma T} e^{j\omega T}$?

  - For s on the imaginary axis, $s = j\omega$, z is on the unit circle $z = e^{j\omega T}$

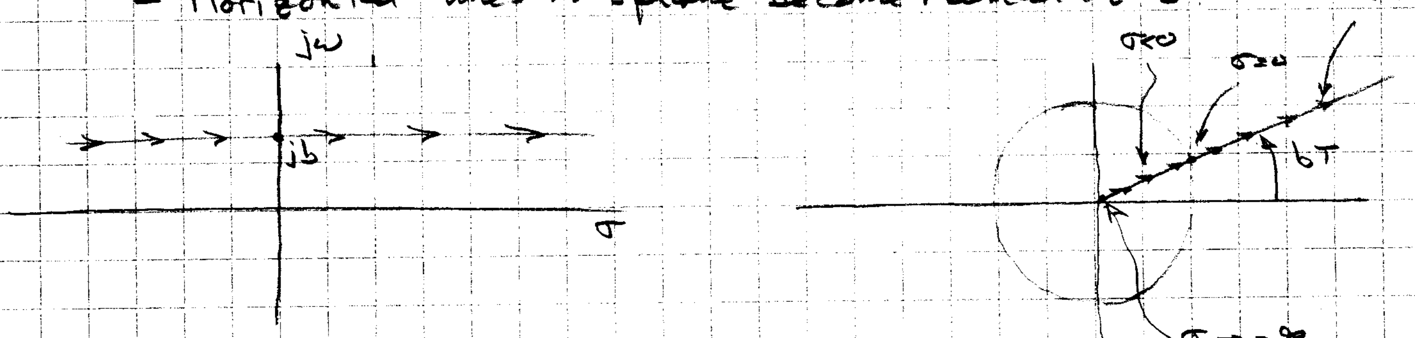


  - For s in RHP $s = a + j\omega$, $a > 0$, we have

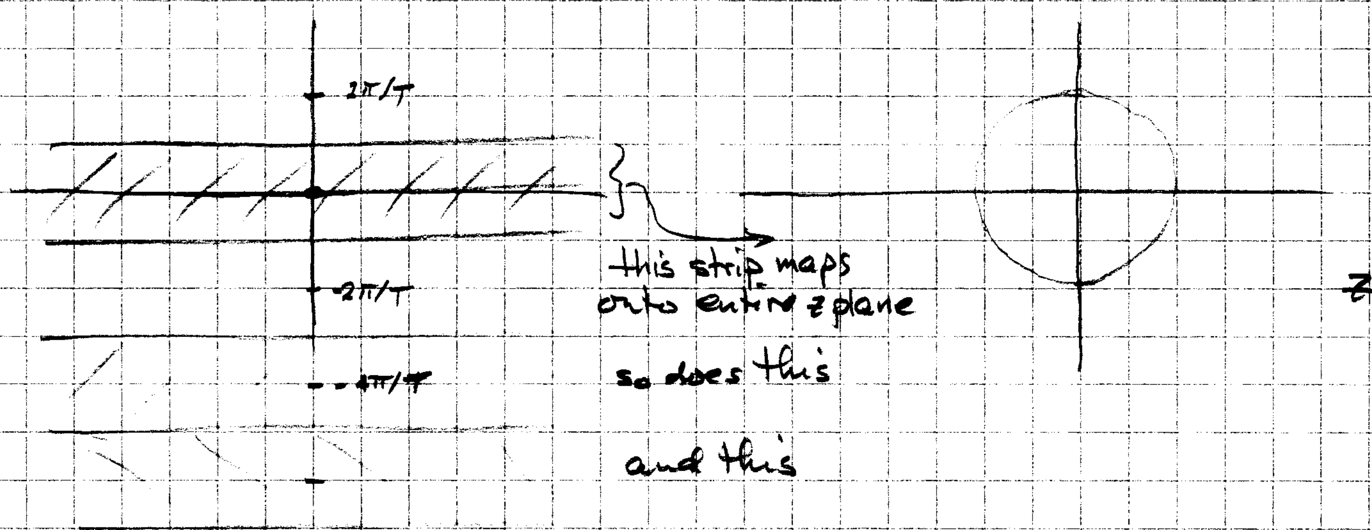


so RHP corresponds to exterior of unit circle, LHP to interior of circle.

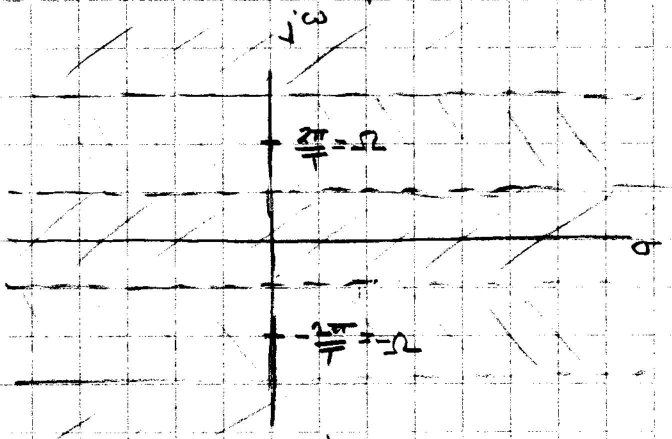
  - Horizontal lines in s plane become radial in z



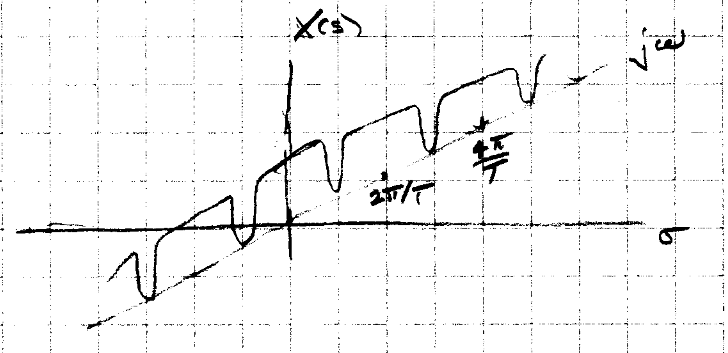
- Poles outside unit circle in z plane correspond to geometrically increasing sequences (analogous to RHP poles in s plane for LT). Similarly, poles inside circle $\rightarrow$ geometric decay.



So on the s-plane $X(s)$ repeats in strips:



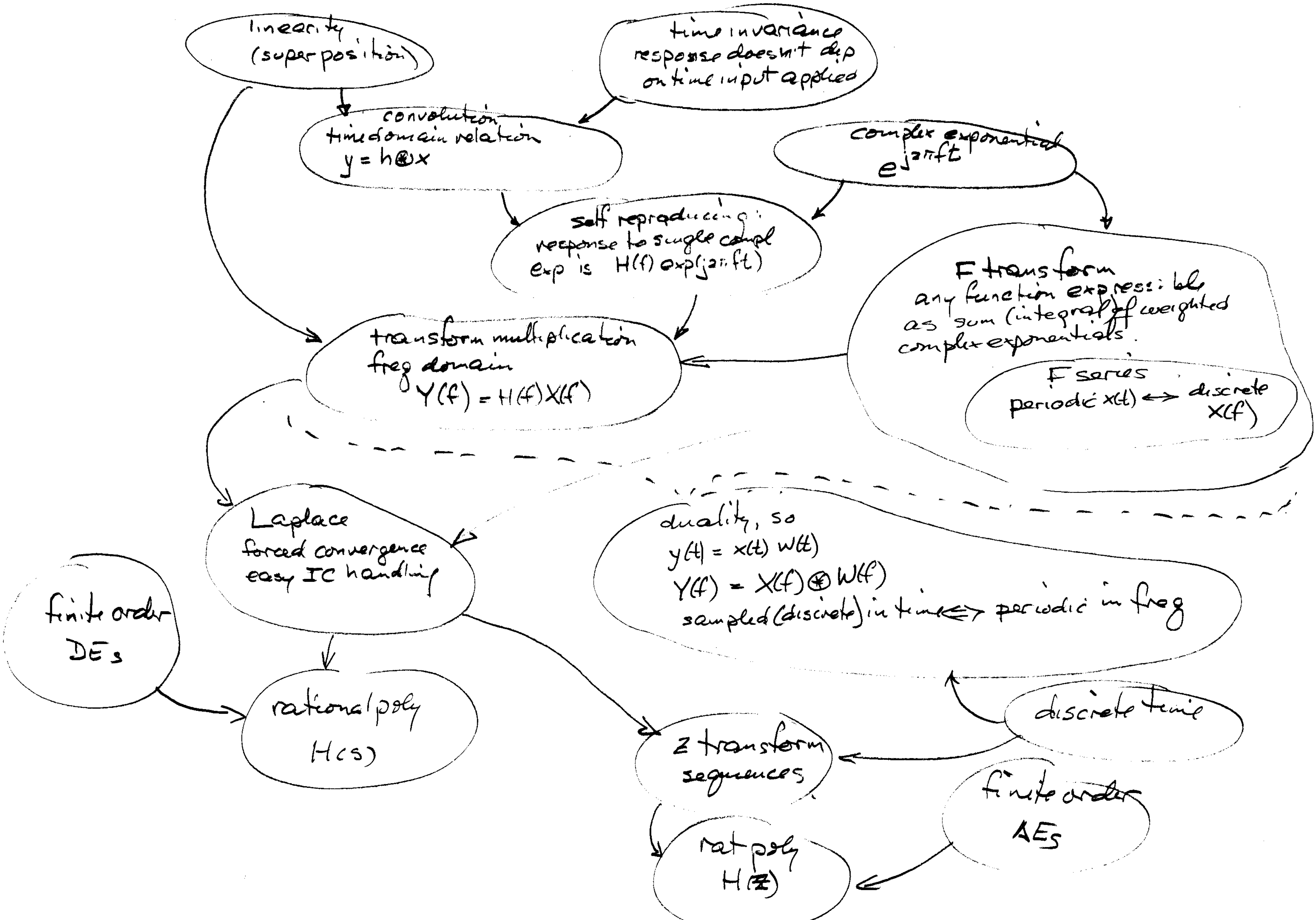
plan view

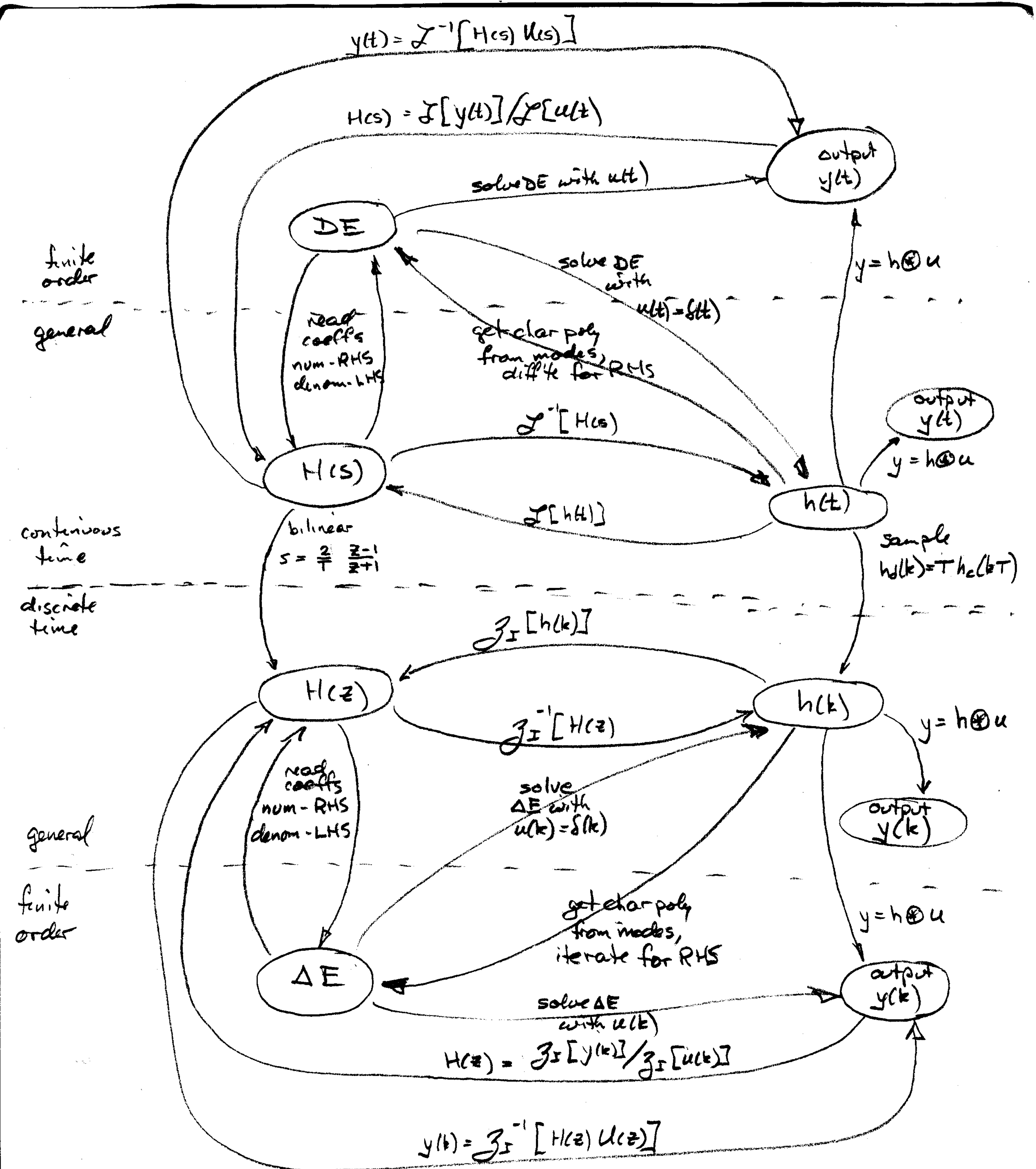


cut along the $j\omega$ axis (hence the FT)

Nothing new here — just the way the LT looks for discrete time.

Logic of this course





y(t) = ℒ⁻¹[H(s) U(s)]
H(s) = ℒ[y(t)]/ℒ[u(t)]
output y(t)
solve DE with u(t)
DE
finite order
general
solve DE with u(t) = δ(t)
y = h ⊛ u
read coeffs num - RHS denom - LHS
get char poly from modes, diff'te for RHS
ℒ⁻¹[H(s)]
output y(t)
y = h ⊛ u
H(s)
h(t)
ℒ[h(t)]
continuous time
bilinear s = 2/T (z-1)/(z+1)
sample h_d(k) = T h_c(kT)
discrete time
𝒵_I[h(k)]
H(z)
h(k)
𝒵_I⁻¹[H(z)]
y = h ⊛ u
read coeffs num - RHS denom - LHS
solve ΔE with u(k) = δ(k)
output y(k)
general
finite order
get char poly from modes, iterate for RHS
y = h ⊛ u
ΔE
solve ΔE with u(k)
output y(k)
H(z) = 𝒵_I[y(k)]/𝒵_I[u(k)]
y(k) = 𝒵_I⁻¹[H(z) U(z)]

$$\begin{tikzcd}
x(t) \arrow[rr, bend left, "\mathcal{F}[x(t)]"] & & X(f) \arrow[ll, bend left, "\mathcal{F}^{-1}[X(f)]"]
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\text{periodic}\\ x(t)\end{array} \arrow[r, bend left, "x_i = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi i t/T}\, dt"] & \begin{array}{c} x(t) = \sum\limits_{i=-\infty}^{\infty} X_i\, e^{j2\pi i t/T} \\ \text{periodic}\end{array} \arrow[r, bend left, "\mathcal{F}[x(t)]"] & \begin{array}{c} X(f) = \sum\limits_{i} X_i\, \delta(f - i/T) \\ \text{discrete, equispaced}\end{array} \arrow[l, bend left, "\mathcal{F}^{-1}[X(f)]"]
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c} x(t) = \sum\limits_{k=-\infty}^{\infty} x_k\, \delta(t - k/W) \\ \text{discrete, equispaced}\end{array} \arrow[r, bend right, "\mathcal{F}[x(t)]"'] & \begin{array}{c} X(f) = \sum\limits_{k=-\infty}^{\infty} x_k\, e^{-j2\pi k f/W} \\ \text{periodic}\end{array} \arrow[l, bend right, "\mathcal{F}^{-1}[X(f)]"'] & \begin{array}{c}\text{periodic}\\ X(f)\end{array} \arrow[l, bend right, "x_k = \frac{1}{W}\int_{-W/2}^{W/2} X(f)\, e^{j2\pi k f/W}\, df"']
\end{tikzcd}$$