SIMON FRASER UNIVERSITY
School of Engineering Science
ENSC 380 Linear Systems

COMPLEX EXPONENTIALS AND ORTHOGONALITY

1. GRAPHING COMPLEX FUNCTIONS

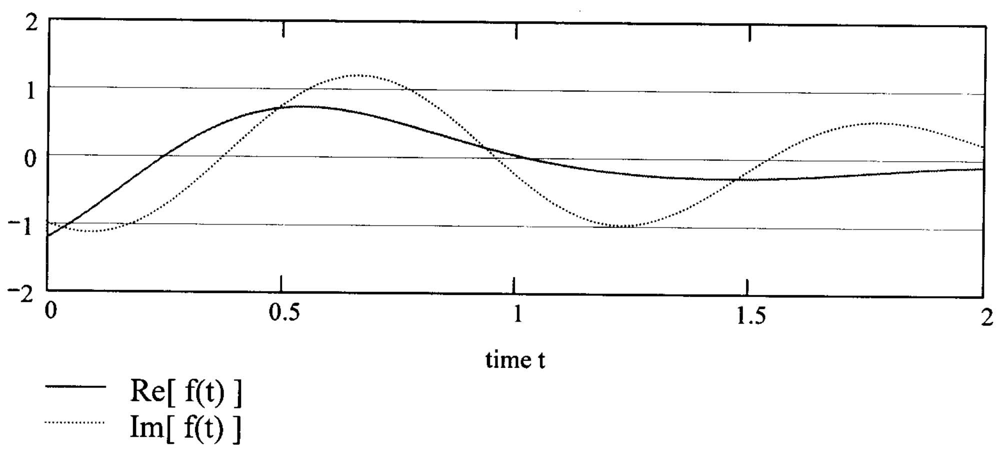
Complex numbers contain two real numbers: the real part and the imaginary part (or the magnitude and phase). Similarly, complex functions of a real variable like time contain two real functions. We can write the function complex $f(t)$ as

$$f(t) = \mathrm{Re}(f(t)) + j \cdot \mathrm{Im}(f(t)) \qquad \mathrm{Re}[f(t)] \text{ and } \mathrm{Im}[f(t)] \text{ are both real}$$

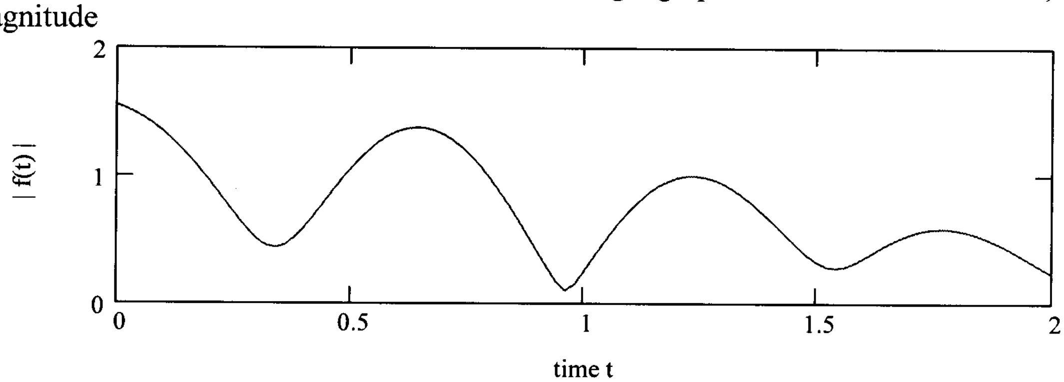
or

$$f(t) = |f(t)| \cdot e^{j \cdot \arg(f(t))} \qquad |f(t)| \text{ and } \arg(f(t)) \text{ are both real}$$

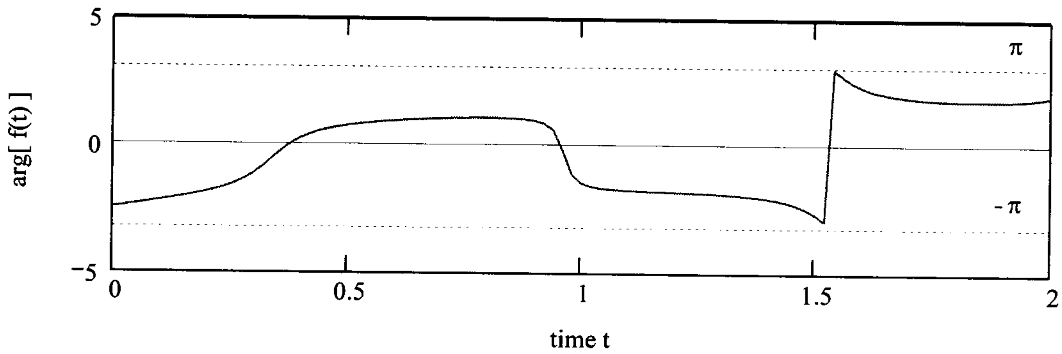
Graphing a complex function is a little different from graphing a real function. We could, for example, plot the real and imaginary parts together, both as functions of time:



Alternatively, we could plot magnitude and phase as functions of time. They have different units, so it's less likely we would see them on a single graphs. For the same function, here's magnitude

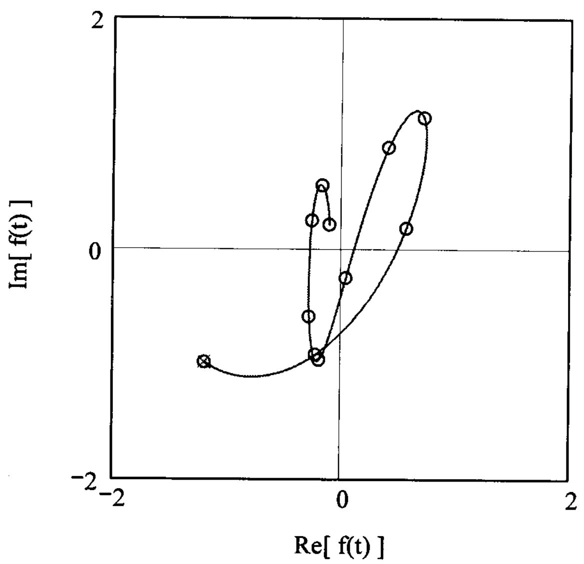


and here's phase.



Note that arg($f(t)$) has the range [$-\pi$,π), so there can be abrupt jumps as the phase crosses π or $-\pi$. Also there can be abrupt reversals in phase as the complex signal cruises past the origin (indicated by near-nulls in the magniude).

There is another way to plot complex functions, though, and it is often more meaningful. A parametric plot puts Re[$f(t)$] and Im[$f(t)$] on the two axes, so that they are both parameterized by time. For the same function that we saw in the previous graphs, we have



The little dots are equispaced in time, to give an idea of speed along the trajectory. They are entirely optional. The first one has a special mark, so you know which way to trace the curve.

When the signal skims past the origin, we see a simultaneous near- null in the magnitude plot and a jump of π in the phase plot.

All three plots have their uses. Plotting real and imaginary parts doesn't usually reveal interesting things, although that formulation is very helpful in analysis. Magnitude and phase plots are already familiar to you, when the domain is frequency ω, instead of time, and you are plotting frequency response. The parametric plots often show characteristics of the function more clearly, but they don't show rate of change very well. Now go over the various plots of the function $f(t)$ and try to relate features in one plot type with those in another.

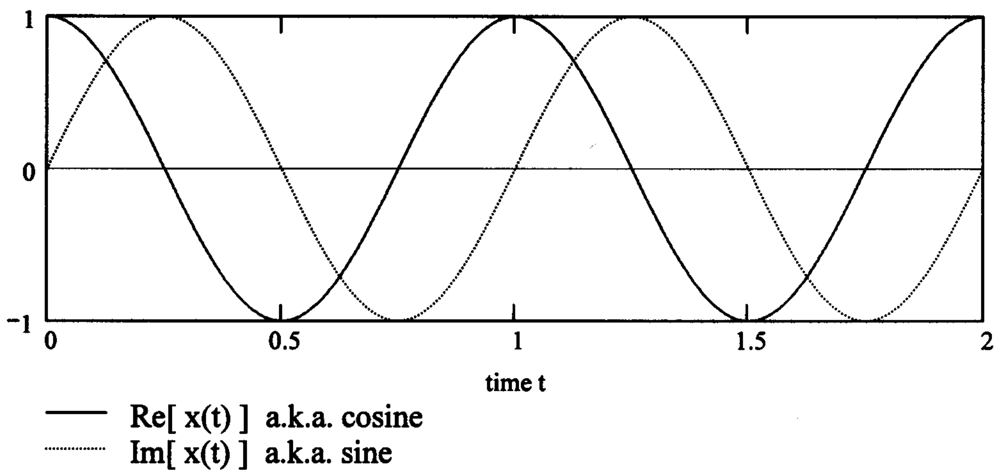
The function used for these plots was generated randomly. **If you'd like to see others, go to the menu and click Math/Calculate Worksheet.**

2 COMPLEX EXPONENTIALS

You already know that $e^{j\theta}=\cos(\theta)+j\sin(\theta)$ for any choice of angle θ. If θ increases linearly with time as $\theta=\omega\cdot t=2\cdot\pi\cdot f\cdot t$, then we define the function

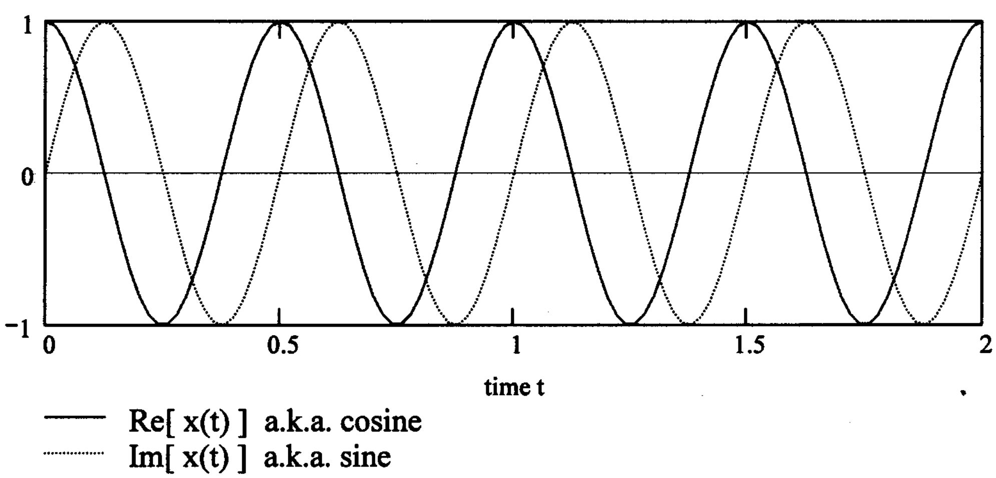
$$x(t)=e^{j\cdot\omega\cdot t}=e^{j\cdot 2\cdot\pi\cdot f\cdot t}$$

as the complex exponential, where ω is the frequency in rad/s and *f* is the frequency in Hz. We can plot the function as



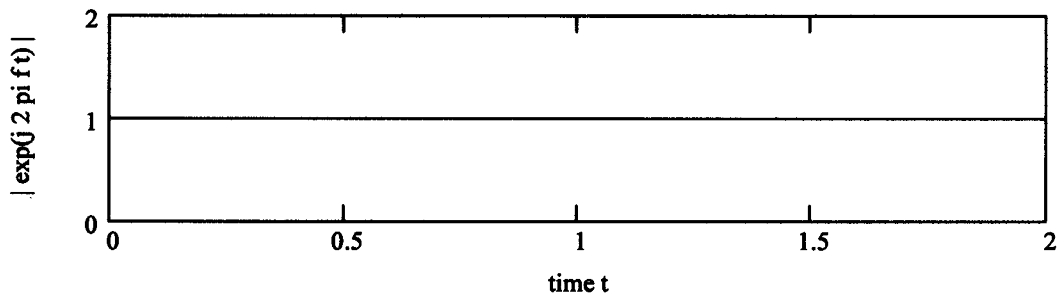
Complex Exponential, 1 Hz

and if the frequency increases, it's



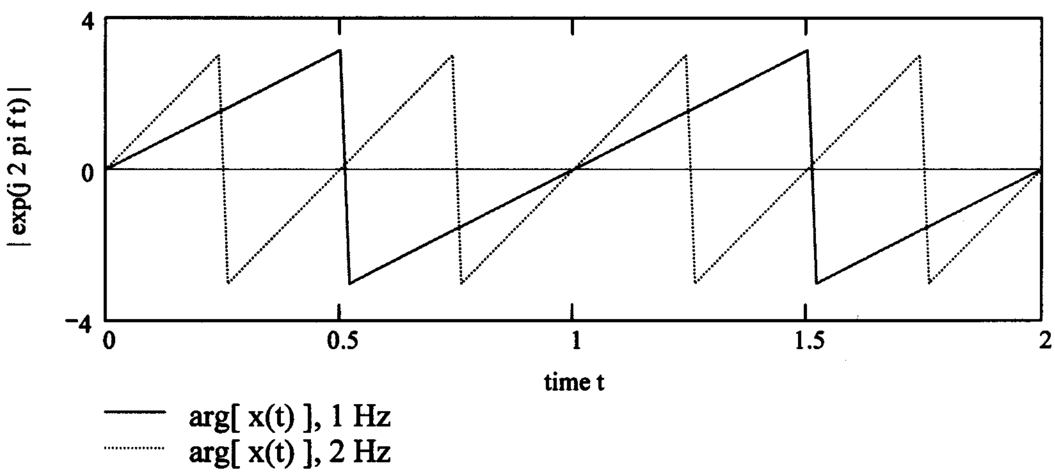
Complex Exponential, 2 Hz

The magnitude and phase plot is pretty straightforward. The magnitude is flat at 1, regardless of frequency.



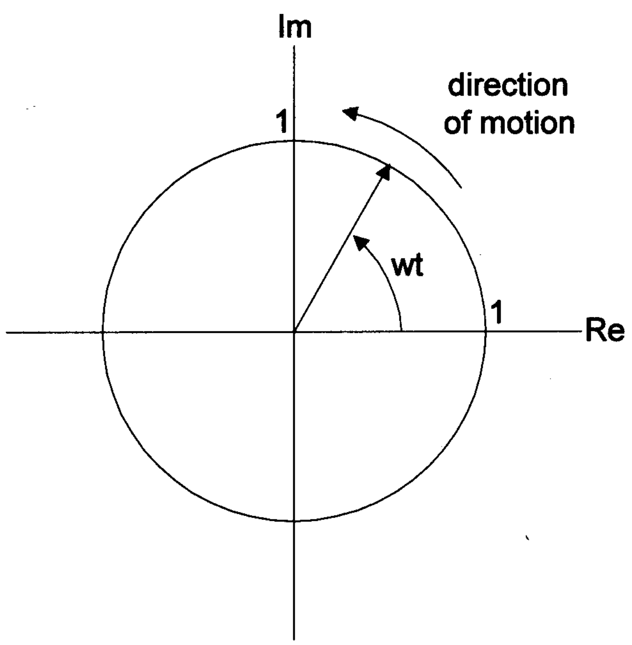
Magnitude of Complex Exp, 1 and 2 Hz

and the phase increases more quickly with higher frequencies:



Phase of Complex Exp, 1 and 2 Hz

The parametric plot is interesting, because the trajectory of $e^{j\omega t} = e^{j2\pi ft}$ is just a progression around the unit circle at constant speed. This makes sense, because its magnitude is always unity, and its phase increases linearly with time.



3. ORTHOGONALITY OF FUNCTIONS

We say that two vectors are orthogonal if their inner product (the dot product) is zero:

$$\mathbf{x} \bullet \mathbf{y} = \sum_{i=1}^{N} x_i \cdot y_i = 0$$

If the components of the vectors were samples of functions taken at equispaced points in time we could write the inner product as

$$\mathbf{x} \bullet \mathbf{y} = \sum_{i=1}^{N} x(t_i) \cdot y(t_i) = 0$$

From here, it's just a small step to a definition of inner product and orthogonality for functions of continuous time: two functions are orthogonal over $[0,T)$ if

$$\int_0^T x(t) \cdot y(t)\, dt = 0$$

Here are some examples. The first is straightforward: $x(t)=1$ and $y(t)=t$ are orthogonal over [-1,1], since

$$\int_{-1}^{1} x(t) \cdot y(t)\, dt = \int_{-1}^{1} 1 \cdot t\, dt = 0$$

The second example is the sine and cosine of a given frequency $1/T$. They are orthogonal over any multiple of the period T; that is,

$$\int_0^{K \cdot T} \cos\left(2 \cdot \pi \cdot \frac{t}{T}\right) \cdot \sin\left(2 \cdot \pi \cdot \frac{t}{T}\right) dt = \int_0^{K \cdot T} \frac{1}{2} \cdot \sin\left(4 \cdot \pi \cdot \frac{t}{T}\right) dt = 0$$

We didn't have to start the integral at 0, either; they are orthogonal over any interval of length equal to a multiple of the period:

$$\int_{t_0}^{t_0 + K \cdot T} \cos\left(2 \cdot \pi \cdot \frac{t}{T}\right) \cdot \sin\left(2 \cdot \pi \cdot \frac{t}{T}\right) dt = 0$$

The first two graphs of Section 2 show pictorially why they are orthogonal: the product of sine and cosine oscillates between positive and negative values, and nets out to zero over one period. The third example is sine and cosines with frequencies at integer multiples of $1/T$. Suppose

$$x(t)=\cos\left(2\cdot\pi\cdot\frac{m}{T}\cdot t\right) \qquad y(t)=\cos\left(2\cdot\pi\cdot\frac{n}{T}\cdot t\right)$$

Then the inner product over T is

$$\int_0^T x(t)\cdot y(t)\,dt=\int_0^T \frac{1}{2}\cdot\cos\left[2\cdot\pi\cdot\left(\frac{m-n}{T}\right)\cdot t\right]+\frac{1}{2}\cdot\cos\left[2\cdot\pi\cdot\left(\frac{m+n}{T}\right)\cdot t\right]dt$$

The second term always integrates to zero. The first term integrates to zero if m and n are different, and to $T/2$ if $m=n$. Therefore, the functions are orthogonal if their frequencies differ by integer multiples of $1/T$. Again, we could have taken the integral over any interval $[t_o,t_o+KT)$. The conclusion is also the same if either or both of $x(t)$ and $y(t)$ are sin, instead of cos.

4. APPLICATION TO FOURIER SERIES

If a periodic function $x(t)$ is written as a Fourier series in sines and cosines, it is

$$x(t)=A_0+\sum_{i=1}^{\infty} A_i\cdot\cos\left(2\cdot\pi\cdot\frac{i}{T}\cdot t\right)+\sum_{i=1}^{\infty} B_i\cdot\sin\left(2\cdot\pi\cdot\frac{i}{T}\cdot t\right)$$

where the A_i and B_i coefficients are all real, and the sum is taken only over positive frequencies. If we know $x(t)$, and want to determine coefficient A_7, say, all we have to do is this:

$$\int_0^T x(t)\cdot\cos\left(2\cdot\pi\cdot\frac{7}{T}\cdot t\right)dt=\frac{T}{2}\cdot A_7$$

where the equality follows from the orthogonality of the cosines and sines. Hence the coefficient is

$$A_7=\frac{2}{T}\cdot\int_0^T x(t)\cdot\cos\left(2\cdot\pi\cdot\frac{7}{T}\cdot t\right)dt$$

The sine coefficients B_i are determined similarly. The dc term has no factor of 2:

$$A_0=\frac{1}{T}\cdot\int_0^T x(t)\,dt$$

5. ORTHOGONALITY OF COMPLEX EXPONENTIALS

For complex time functions, the definition of inner product requires conjugating the second one. Consequently, two complex functions of time are orthogonal over [0,*T*) if

$$\int_0^T x(t)\cdot\overline{y(t)}\,dt = 0$$

We could also use this definition with real functions, of course, since the conjugate operation leaves them unchanged.

Now we'll apply the definition to complex exponentials with frequencies at multiples of $1/T$

$$x(t) = e^{j\cdot 2\cdot\pi\cdot\frac{m}{T}\cdot t} \qquad y(t) = e^{j\cdot 2\cdot\pi\cdot\frac{n}{T}\cdot t}$$

Their inner product is

$$\int_0^T x(t)\cdot\overline{y(t)}\,dt = \int_0^T e^{j\cdot 2\cdot\pi\cdot\frac{m}{T}\cdot t}\cdot e^{-j\cdot 2\cdot\pi\cdot\frac{n}{T}\cdot t}\,dt = \int_0^T e^{j\cdot 2\cdot\pi\cdot\frac{m-n}{T}\cdot t}\,dt$$

The result is zero if ***m*** and ***n*** differ, and *T* if they are equal. It's perhaps easier to see this if we fall back to sines and cosines:

$$\int_0^T e^{j\cdot 2\cdot\pi\cdot\frac{m-n}{T}\cdot t}\,dt = \int_0^T \cos\left(2\cdot\pi\cdot\frac{m-n}{T}\cdot t\right)dt + j\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{m-n}{T}\cdot t\right)dt$$

Clearly, if ***m*** and ***n*** differ, the integral over an integer number of periods is zero. If ***m*** and ***n*** are equal, then the second integrand is zero, and the first is 1. In summary, the inner products of these complex exponentials are zero if the frequencies differ and *T* if the frequencies are the same.

6. APPLICATION TO COMPLEX FOURIER SERIES

A periodic function *x*(*t*) can be written as a complex Fourier series

$$x(t) = \sum_{i=-\infty}^{\infty} X_i\cdot e^{j\cdot 2\cdot\pi\cdot\frac{i}{T}\cdot t}$$

where X_i is the complex coefficient at frequency i/T and the sum is over all frequencies, positive and negative. This is a lot more compact that the sine and cosine series of Section 4. It is also equivalent, if the coefficients for a frequency and its negative counterpart are complex conjugates. For example, consider the terms at $+7/T$ and $-7/T$ and pair them in the sum:

$$X_{-7}\cdot e^{-j\cdot 2\cdot\pi\cdot\frac{7}{T}\cdot t} + X_7\cdot e^{j\cdot 2\cdot\pi\cdot\frac{7}{T}\cdot t}$$

If $X_{-7}=\overline{X_7}$ (this is the "conjugate symmetry" condition) then we can write the sum as

$$\overline{X_7}\cdot e^{-j\cdot 2\cdot\pi\cdot\frac{7}{T}\cdot t} + X_7\cdot e^{j\cdot 2\cdot\pi\cdot\frac{7}{T}\cdot t} = 2\cdot\mathrm{Re}\left(X_7\cdot e^{j\cdot 2\cdot\pi\cdot\frac{7}{T}\cdot t}\right)$$

$$= 2\cdot\mathrm{Re}\left(X_7\right)\cdot\cos\left(2\cdot\pi\cdot\frac{7}{T}\cdot t\right) - 2\cdot\mathrm{Im}\left(X_7\right)\cdot\sin\left(2\cdot\pi\cdot\frac{7}{T}\cdot t\right)$$

$$= A_7\cdot\cos\left(2\cdot\pi\cdot\frac{7}{T}\cdot t\right) + B_7\cdot\sin\left(2\cdot\pi\cdot\frac{7}{T}\cdot t\right)$$

where the relation between the *A*'s and *B*'s and the *X*'s is clear from the last two lines. The complex Fourier series is therefore equivalent to the sine and cosine series if conjugate symmetry holds.

In order to determine the coefficients from $x(t)$, we do this

$$\int_0^T x(t)\cdot e^{-j\cdot 2\cdot\pi\cdot\frac{k}{T}\cdot t}\,dt = \int_0^T e^{-j\cdot 2\cdot\pi\cdot\frac{k}{T}\cdot t}\cdot\sum_{i=-\infty}^{\infty} X_i\cdot e^{j\cdot 2\cdot\pi\cdot\frac{i}{T}\cdot t}\,dt$$

$$= \sum_{i=-\infty}^{\infty} X_i\cdot\int_0^T e^{j\cdot 2\cdot\pi\cdot\frac{i-k}{T}\cdot t}\,dt = X_k\cdot T$$

where the last equality follows from the orthogonality demonstrated in the last Section. Hence the coefficients are determined simply from

$$X_k = \frac{1}{T}\cdot\int_0^T x(t)\cdot e^{-j\cdot 2\cdot\pi\cdot\frac{k}{T}\cdot t}\,dt$$