

SIMON FRASER UNIVERSITY
School of Engineering Science

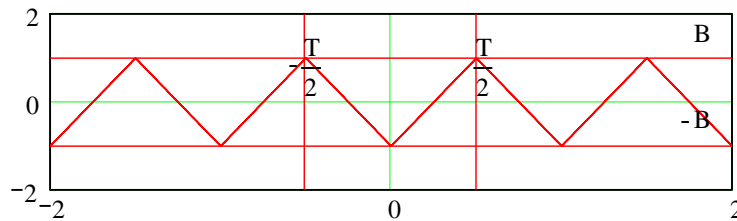
ENSC 380 Linear Systems

EXAMPLES OF FOURIER SERIES

The examples below have been selected to illustrate several properties and calculation methods related to Fourier series. For the theoretical development, refer to your class notes. Regions highlighted in yellow indicate parameter values you can change.

1. ANALYTICAL DETERMINATION OF COEFFICIENTS

Occasionally, you find situations in which you have the mathematical form of the periodic function $x(t)$ and you can calculate the coefficients analytically. Here, we'll do the triangle wave with amplitude B and period T . It looks like this for $B := 1$ and $T := 1$:



You can choose any interval of length T for calculation of the coefficients. I'll use $[-T/2, T/2)$ because the expressions for $x(t)$ are easy

$$x(t) = B \left(-1 - 4 \cdot \frac{t}{T} \right) \quad \text{for} \quad -\frac{T}{2} \leq t < 0 \quad \text{and} \quad x(t) = B \left(-1 + 4 \cdot \frac{t}{T} \right) \quad \text{for} \quad -0 \leq t < \frac{T}{2}$$

In the complex Fourier analysis, the coefficients are determined by

$$X_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} dt$$

$$= \frac{B}{T} \int_{-\frac{T}{2}}^0 \left(-1 - 4 \cdot \frac{t}{T} \right) \cdot e^{-j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} dt + \frac{B}{T} \int_0^{\frac{T}{2}} \left(-1 + 4 \cdot \frac{t}{T} \right) \cdot e^{-j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} dt$$

Tedious integration by parts gives $X_n = 2 \cdot B \cdot \frac{[-1 + (-1)^n]}{\pi^2 \cdot n^2}$ or

$$X_n = 0 \quad \text{for even } n, \text{ and } X_n = \frac{-4 \cdot B}{\pi^2 \cdot n^2} \quad \text{for odd } n$$

Even harmonics are zero, just as in the 50% duty cycle square wave, and for the same reason - symmetry. An asymmetric square wave or triangle wave has all harmonics. It's also interesting that the amplitude (magnitude) of the coefficients decreases as $1/n^2$, in contrast to the square wave, where they decrease as $1/n$. You'll see the reason in the next section.

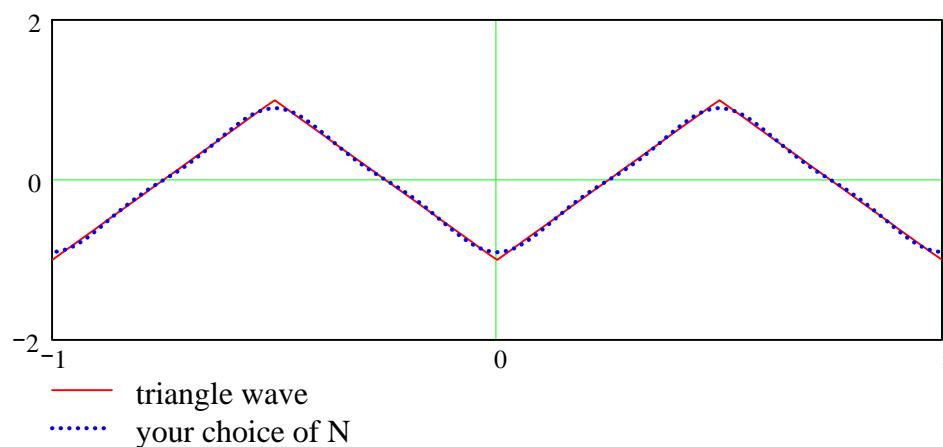
Because $x(t)$ is a real function, the coefficients should be conjugate symmetric:

$X_{-n} = \overline{(X_n)}$. Are they? Here the coefficient at $-n$ has the same value as the one at n , and they are both real, so conjugation has no effect. Consequently, they are conjugate symmetric.

That was a lot of work. Let's see how good the series is as the number of terms increases. The Fourier synthesis sum with a finite number of terms ($-N..N$) and parameters B and T is

$$x_{\text{hat}}(t, N, B, T) := \sum_{n=-N}^N 2 \cdot B \cdot \frac{[-1 + (-1)^n]}{\pi^2 \cdot n^2} \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t}$$

N := 3 <== your choice B = 1 T = 1 <== these values set earlier of N



Triangle Wave and Approximations x_{hat}

The series converges rapidly, with most of the error at the peaks of the triangle wave, where there is a slope discontinuity.

2. DERIVATIVE OF A FOURIER SERIES

As a check on our work in Section 1, we could differentiate $x(t)$. The result is a square wave with amplitude $A=4B/T$, as you can see by looking at the graph at the start of Section 1. Next, we differentiate the Fourier synthesis sum for the triangle and see if the coefficients are those of a square wave:

$$\begin{aligned}
 y(t) &= \frac{d}{dt} x(t) = \frac{d}{dt} \sum_{n=-\infty}^{\infty} 2 \cdot B \cdot \frac{[-1 + (-1)^n]}{\pi^2 \cdot n^2} \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} \\
 &= \sum_{n=-\infty}^{\infty} 2 \cdot B \cdot \frac{[-1 + (-1)^n]}{\pi^2 \cdot n^2} \cdot j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} = \sum_{n=-\infty}^{\infty} 4 \cdot \frac{B}{T} \cdot \frac{j \cdot [-1 + (-1)^n]}{\pi \cdot n} \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t}
 \end{aligned}$$

Substitution for the amplitude gives

$$y(t) = \sum_{n=-\infty}^{\infty} A \cdot \frac{j \cdot [-1 + (-1)^n]}{\pi \cdot n} \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t}$$

which is the Fourier series for a square wave of amplitude A .

We can obtain the same result by considering differentiation to be the result of a linear filter with frequency response $H(f)=j2\pi f$. If the input is a triangle wave, we can use superposition to calculate the response to its Fourier series representation. Component n is a complex exponential at frequency n/T . It gets scaled by the "complex gain" $H(n/T)=j2\pi n/T$. Hence the Fourier coefficient of the output $y(t)$ (i.e., the derivative) at frequency n/T is

$$Y_n = X_n \cdot j \cdot 2 \cdot \pi \cdot \frac{n}{T} = 2 \cdot B \cdot \frac{[-1 + (-1)^n]}{\pi^2 \cdot n^2} \cdot j \cdot 2 \cdot \pi \cdot \frac{n}{T} = 4 \cdot \frac{B}{T} \cdot \frac{j \cdot [-1 + (-1)^n]}{\pi \cdot n}$$

which is precisely what we derived above.

Are these coefficients conjugate symmetric? Replace n by $-n$ and the sign of Y_n changes. It's purely imaginary, so this *is* conjugation. Hence conjugate symmetric, as required by the fact that $y(t)$ is always real.

3. EQUIVALENCE OF COMPLEX AND SINE/COSINE SERIES

The sine/cosine Fourier series is often used instead of the complex exponential series. It's bulkier and you have to remember some trigonometric identities, but it uses only positive frequencies and you don't have to use complex arithmetic. It looks like this:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\left(2 \cdot \pi \cdot \frac{n}{T} \cdot t\right) + \sum_{n=1}^{\infty} B_n \cdot \sin\left(2 \cdot \pi \cdot \frac{n}{T} \cdot t\right)$$

The two representations are equivalent for real functions $x(t)$. To see this, rewrite the complex Fourier synthesis sum, pairing positive and negative terms:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} = X_0 + \left[\sum_{n=1}^{\infty} \left(X_{-n} \cdot e^{-j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} + X_n \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} \right) \right]$$

Because of conjugate symmetry $X_{-n} = \overline{(X_n)}$, the two terms are complex conjugates. Hence

$$\begin{aligned} x(t) &= X_0 + \sum_{n=1}^{\infty} 2 \cdot \operatorname{Re} \left(X_n \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} \right) \\ &= X_0 + \sum_{n=1}^{\infty} 2 \cdot \operatorname{Re}(X_n) \cdot \cos\left(2 \cdot \pi \cdot \frac{n}{T} \cdot t\right) - \sum_{n=1}^{\infty} 2 \cdot \operatorname{Im}(X_n) \cdot \sin\left(2 \cdot \pi \cdot \frac{n}{T} \cdot t\right) \end{aligned}$$

which has the form of the sine/cosine series with $A_n = 2 \cdot \operatorname{Re}(X_n)$ and $B_n = -2 \cdot \operatorname{Im}(X_n)$.

As an example, calculate the sine/cosine series coefficients for the triangle wave.

$$A_n = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot \cos\left(2 \cdot \pi \cdot \frac{n}{T} \cdot t\right) dt$$

$$= \frac{2 \cdot B}{T} \cdot \int_{-\frac{T}{2}}^0 \left(-1 - 4 \cdot \frac{t}{T}\right) \cdot \cos\left(2 \cdot \pi \cdot \frac{n}{T} \cdot t\right) dt + \frac{2 \cdot B}{T} \cdot \int_0^{\frac{T}{2}} \left(-1 + 4 \cdot \frac{t}{T}\right) \cdot \cos\left(2 \cdot \pi \cdot \frac{n}{T} \cdot t\right) dt$$

$$= 4 \cdot B \cdot \frac{[-1 + (-1)^n]}{\pi^2 \cdot n^2}$$

after integration by parts.

This agrees with our results from Section 1. Similarly, the B coefficients are (another integral)

$$B_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot \sin\left(2\pi \cdot \frac{n}{T} \cdot t\right) dt = 0$$

where another integration by parts was required, only to find that they are all zero. Anything left? Oh, yes, the component at $n=0$. It has a different scale factor

$$A_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt = 0$$

This cumbersome and bulky form of the Fourier series makes the concise form of the complex Fourier series look quite attractive.

4. NUMERICAL DETERMINATION OF COEFFICIENTS

Frequently, you have the mathematical form of the periodic $x(t)$, but evaluating the integrals analytically is either not possible or not worth the effort. In such cases, a numerical integration is appropriate. You could use Simpson's rule, Gaussian quadrature, or whatever your math software provides.

As an example, we'll look at FM synthesis, which is frequently used in music synthesisers because it is an easy way to generate lots of harmonics. You simply take the sine wave output of a VCO (voltage controlled oscillator) and modulate its frequency or phase by another sine wave at a different frequency N times lower. The result is

$$x(t) = \cos\left(2\pi \cdot f_1 \cdot t + \phi(t)\right) \quad \text{where the phase is} \quad \phi(t) = \beta \cdot \cos\left(2\pi \cdot \frac{f_1}{N} \cdot t\right)$$

giving

$$x(t) = \cos\left(2\pi \cdot f_1 \cdot t + \beta \cdot \cos\left(2\pi \cdot \frac{f_1}{N} \cdot t\right)\right)$$

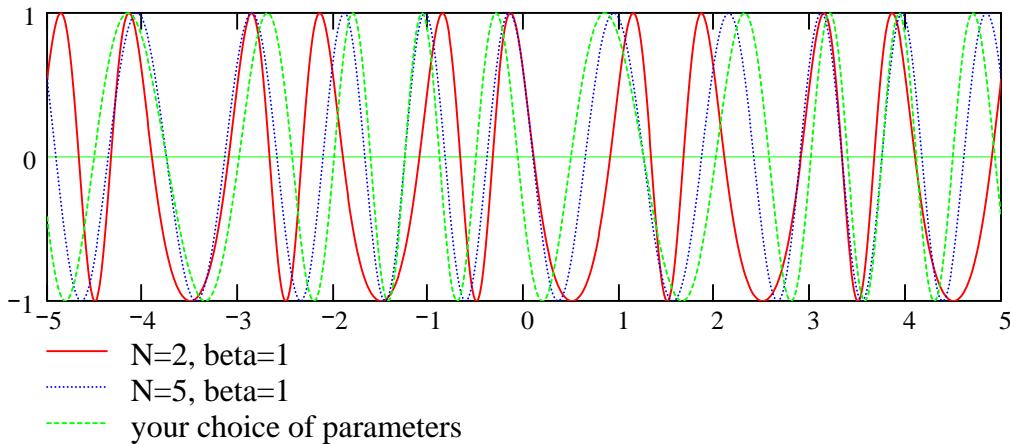
It is periodic with period $T=N/f_1$, so it can be represented as a Fourier series. Calculating the coefficients isn't much fun, although there is a closed form involving Bessel functions. I'll do it numerically.

Define the signal with parameters as

$$x(t, f_1, N, \beta) := \cos\left(2 \cdot \pi \cdot f_1 \cdot t + \beta \cdot \cos\left(2 \cdot \pi \cdot \frac{f_1}{N} \cdot t\right)\right)$$

and, to keep things simple, I'll take $f_1 := 1$ Hz (the results scale to other frequencies). First, look at the signal

$t := -5, -4.99..5$ $N := 5$ $\beta := 2$ \Leftarrow your choice of parameters



Note that the traces on the graph have different periods.

We'll work out the coefficients for $N := 2$, causing the harmonics to be spaced by $\frac{f_1}{N} = 0.5$ Hz. Also choose β below.

$T := \frac{N}{f_1}$ $\beta := 1.5$ $TOL := 10^{-7}$ (error tolerance on integration)

$$X(n) := \frac{1}{T} \int_0^T x(t, f_1, N, \beta) \cdot e^{-j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} dt$$

in Mathcad, this causes the integral to be performed numerically

For non-negative n , we'll save the coefficients in an array. Coefficients for negative n can be obtained by conjugation. Only get the first few. Also calculate the corresponding frequencies:

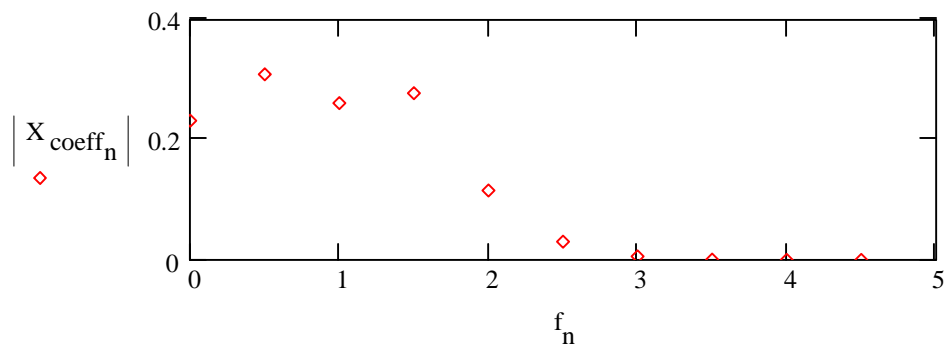
$$n := 0..9 \quad X_{\text{coeff}_n} := X(n) \quad f_n := \frac{f_1}{N} \cdot n$$

And the result is...

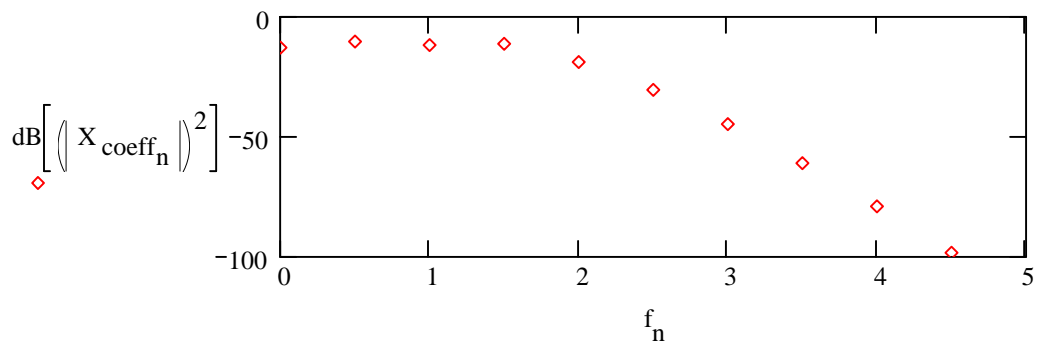
0	
0	-0.232
1	0.309j
2	0.262
3	0.278j
4	-0.116
5	-0.03j
6	$5.885 \cdot 10^{-3}$
7	$8.996j \cdot 10^{-4}$
8	$-1.14 \cdot 10^{-4}$
9	$-1.234j \cdot 10^{-5}$

0	
0	0
1	0.5
2	1
3	1.5
4	2
5	2.5
6	3
7	3.5
8	4
9	4.5

Now let's plot the magnitude (amplitude) of the frequency components of our FM synthesised signal:



or on a dB scale, with $dB(x) := 10 \cdot \log(x)$



Lots of harmonics, all right, even though we used just one sine function and a VCO to build it.

5. FOURIER SERIES AND FILTERING

One of the most useful features of the frequency representation of a signal, such as Fourier series or Fourier transform, is the ease with which filter effects can be calculated. If you have the Fourier components of the input signal X_n and the frequency response $H(f)$ of the filter, then the Fourier components of the output are just

$$Y_n = X_n \cdot H\left(\frac{n}{T}\right)$$

since component n is at frequency n/T and we can apply superposition to calculate the response to each term individually.

As an example, we'll run a square wave with amplitude A and frequency $1/T$ through a simple RC lowpass filter. It's a first order lowpass filter with frequency response

$$H(f) = \frac{1}{1 + j \cdot 2 \cdot \pi \cdot f \cdot \tau} \quad \text{where } \tau \text{ is its time constant.}$$

How to obtain the frequency response is a separate issue, and I'll discuss it briefly below. The input coefficients are, from Section 2,

$$X_n = A \cdot \frac{j \cdot [-1 + (-1)^n]}{\pi \cdot n} \quad \text{or, as a function,} \quad X(n, A, T) := A \cdot \frac{j \cdot [-1 + (-1)^n]}{\pi \cdot n}$$

Hence the output coefficients, also as a function, are

$$Y(n, A, T, \tau) := A \cdot \frac{j \cdot [-1 + (-1)^n]}{\pi \cdot n} \cdot \frac{1}{1 + j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot \tau}$$

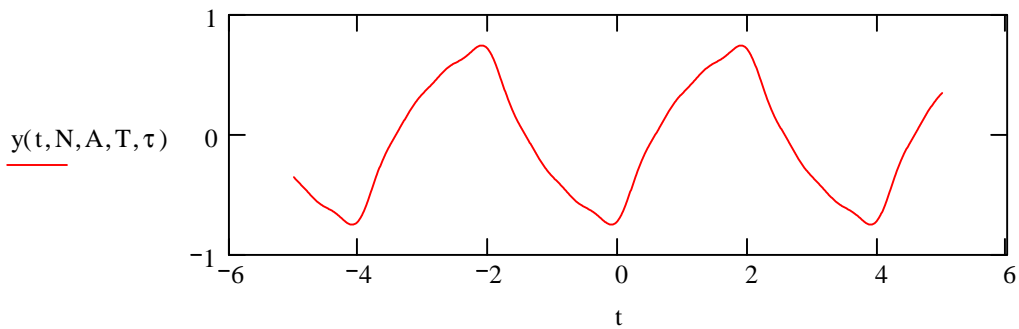
Note this is conjugate symmetric (substitute $-n$ for n and the result is conjugated). Hence $y(t)$ is real.

What does the output look like in time? Just use the Fourier synthesis sum (with a finite number of terms for simplicity):

$$y(t, N, A, T, \tau) := \sum_{n=-N}^N Y(n, A, T, \tau) \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t}$$

I'll plot it below with

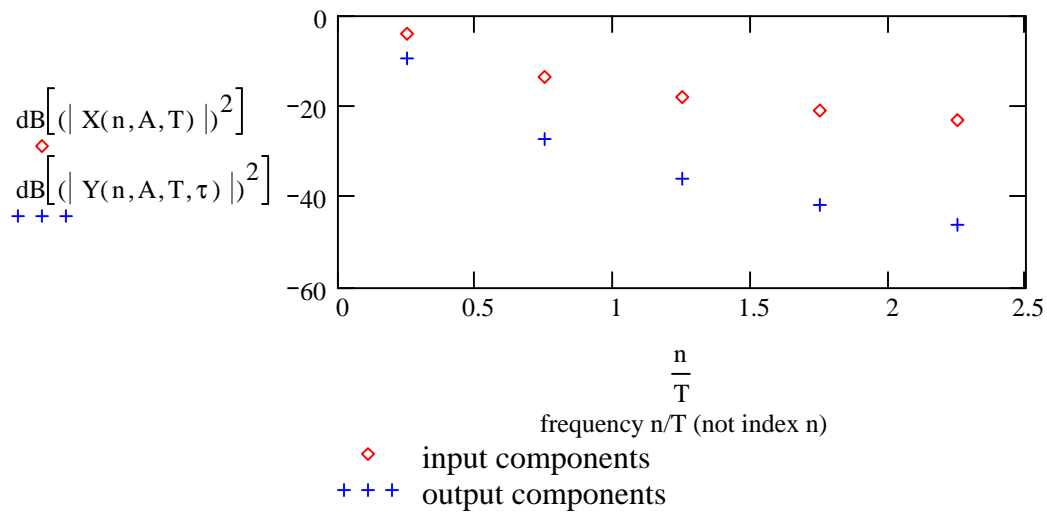
$$N := 8 \quad (\text{so } 2N+1 \text{ terms}) \quad A := 1 \quad T := 4 \quad \tau := 1$$



This certainly looks familiar! Some ripple near the cusps is evident, but it decreases with more terms. Change N above and see.

What is the frequency composition of input and output? Show below is the magnitude in dB for positive frequencies only

$$\text{dB}(x) := \text{if}(x > 10^{-16}, 10 \cdot \log(x), -160)$$



Recall that $T=4$ in this example, so the fundamental frequency is $1/4$. The even harmonics are missing, as expected (they weren't in the input, so why would they be in the output?). The effect of the filter at higher frequencies is clearly evident.

You could also have obtained the filter output by staying in the time domain, and performing a convolution of the square wave input with the impulse response

$$h(t) = \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}} \quad t \geq 0 \quad y(t) = \int_0^{\infty} \text{sq}(t - \alpha) \cdot h(\alpha) d\alpha$$

where $\text{sq}(t)$ denotes the square wave, as a reminder of the shape of $x(t)$. The convolution would have some difficulties with the fact that there is an infinite number of repetitions of the square wave. You can resolve it by summing a geometric series, but it's a nuisance. As an alternative, the sketch solution approach shows you quickly what it looks like, but doesn't give much quantitative information.

Next, how to get the frequency response $H(f)$? Two possibilities come to mind: First, if you have a circuit diagram, just use $j\omega L$, $1/j\omega C$ and R for the impedances, then use standard circuit reduction methods. Second, you could start with the step response, which is easy to remember

$$g(t) = 1 - e^{-\frac{t}{\tau}} \quad \text{Differentiate it to get impulse response} \quad h(t) = \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}}$$

then transform it, as shown in the class notes, to get frequency response:

$$H(f) = \int_0^{\infty} h(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot f \cdot t} dt = \int_0^{\infty} \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}} \cdot e^{-j \cdot 2 \cdot \pi \cdot f \cdot t} dt = \frac{1}{1 + j \cdot 2 \cdot \pi \cdot f \cdot \tau}$$

6. FOURIER ANALYSIS OF EXPERIMENTAL DATA


This case comes up repeatedly. You don't have a mathematical form for $x(t)$, because you recorded it from lab or field measurements. How do you obtain a Fourier decomposition? Simply take N samples equispaced with Δt and approximate the integral with a sum:

$$X_n = \frac{1}{T} \int_0^T x(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot t} dt = \frac{1}{N \cdot \Delta t} \cdot \sum_{i=0}^{N-1} x(i \cdot \Delta t) \cdot e^{-j \cdot 2 \cdot \pi \cdot \frac{n}{T} \cdot i \cdot \Delta t} \cdot \Delta t$$

$$= \frac{1}{N} \cdot \sum_{i=0}^{N-1} x(i \cdot \Delta t) \cdot e^{-j \cdot \frac{2 \cdot \pi}{N} \cdot n \cdot i}$$

This form also goes by the name of the discrete Fourier transform (DFT) and it is usually implemented by the fast Fourier transform (FFT) algorithm.

We'll do an example. Read in measurements of a signal, with some measurement noise, sampled at 100 Hz:

```
x :=  xmeas.txt      N := length(x)      N = 32      i := 0..N - 1
```

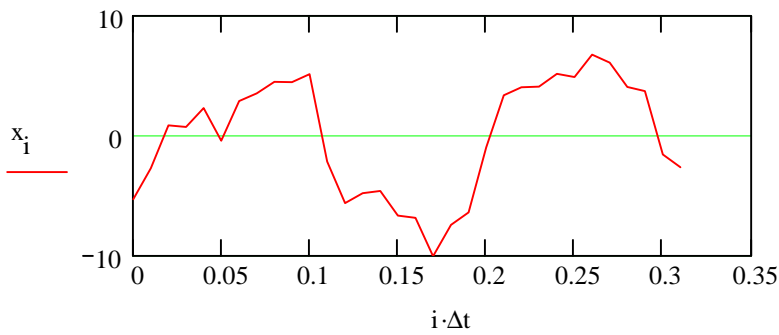
The sampling interval, and the period and fundamental frequency, of the signal are

$$\Delta t := \frac{1}{100 \cdot \text{Hz}} \quad T := N \cdot \Delta t \quad \frac{1}{T} = 3.125 \cdot \text{Hz}$$

and the frequency components are spaced by

$$\frac{1}{T} = 3.125 \cdot \text{Hz}$$

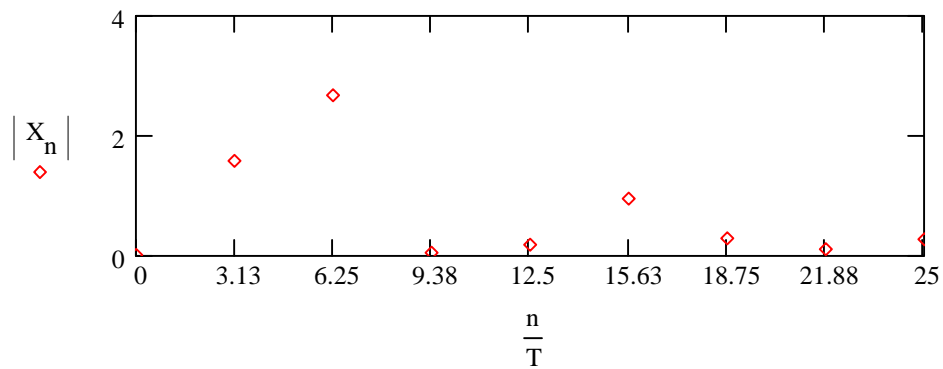
Here is the function:



The first few frequency components are:

$$n := 0 \dots \frac{N}{2} \quad X_n := \frac{1}{N} \cdot \sum_{i=0}^{N-1} x_i \cdot e^{-j \cdot \frac{2 \cdot \pi}{N} \cdot n \cdot i}$$

Let's have a look at their magnitude for positive frequencies:



Fourier analysis has revealed strong components at 1, 2 and 5 times the fundamental frequency of $\frac{1}{T} = 3.125 \cdot \text{Hz}$. The other components may be junk due to the measurement error.

7. DOES PHASE MATTER?

Most of the plots so far have shown only the magnitude of the Fourier coefficients. Does this mean that phase is unimportant? NO. To show its importance, we can do a little experiment. We'll take a few coefficients with the same amplitudes, but different phases, and synthesize the waveform to see if there are any visible differences.

ORIGIN := -4 (Mathcad stuff - don't worry about it)

$m := 0..4$ $A_m := \frac{1}{1+m}$ four coefficients, all amplitude A

Now to create the Fourier coefficients for three different sets of random phases

$U_m := A_m \cdot e^{j \cdot \text{rnd}(2 \cdot \pi)}$ $V_m := A_m \cdot e^{j \cdot \text{rnd}(2 \cdot \pi)}$ $W_m := A_m \cdot e^{j \cdot \text{rnd}(2 \cdot \pi)}$

Make them conjugate symmetric, for a real waveform:

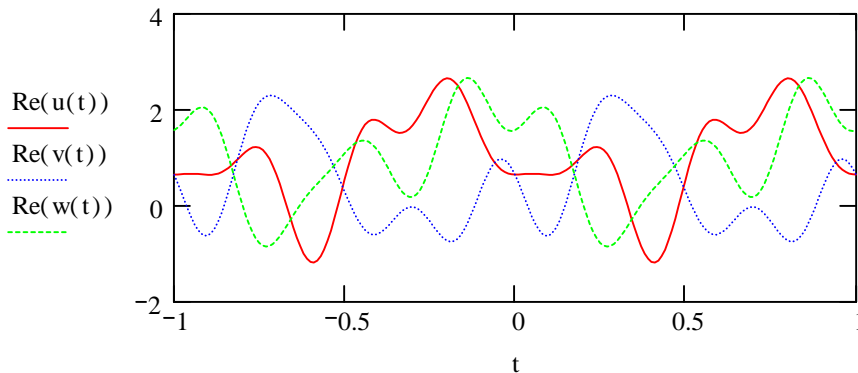
$U_{-m} := \overline{U_m}$ $V_{-m} := \overline{V_m}$ $W_{-m} := \overline{W_m}$

and synthesise

$T := 1$

$u(t) := \sum_{m=-4}^4 U_m \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{m}{T} \cdot t}$ $v(t) := \sum_{m=-4}^4 V_m \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{m}{T} \cdot t}$ $w(t) := \sum_{m=-4}^4 W_m \cdot e^{j \cdot 2 \cdot \pi \cdot \frac{m}{T} \cdot t}$

$t := -T, -0.99 \cdot T .. T$



I used the $\text{Re}()$ operator because roundoff error leaves a small imaginary component, and Mathcad only plots real functions.

There you are. Quite different waveforms, even though the amplitudes of their Fourier coefficients are identical.