SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 320 Electric Circuits II

One-Sided Laplace Transforms

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1. Introduction

You probably have some familiarity with the standard Laplace transform and its inversion by partial fractions. However, you may not have seen its links with the classical theory of complex variables and the role of residues in the inversion of a transform. These notes will fill a little of that gap. They're not a substitute for proper study of complex variables, but they will show you the progression from Fourier series to Laplace transform, the meaning of “region of convergence” and how to invert transforms by residues.

The discussion is restricted to one-sided functions – ones that are zero for negative time. This is commonly the case in control system design or any other situation where initial conditions and transient behaviour after application of an input are important. However, many signals and functions in communications are two-sided, and require the two-sided Laplace transform (which, thankfully, is beyond the scope of this note).

2. Definition of One-Sided Transform

A one-sided function is zero for negative time; that is, $t < 0^-$, where 0^- denotes a time just before 0 (this formulation makes allowance for impulses at time zero, $\delta(t)$. For exponential, sinusoidal and polynomial signals, and for systems described by linear differential equations with constant coefficients, the Laplace transform provides a convenient simplification. It’s a way of expressing any function as a superposition (integral) of complex exponentials.

The Laplace transform of a function $x(t)$ is

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

where s is a complex variable. The inverse transform is given by

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} dt$$

where c is a real constant selected for convergence of $X(s)$, as discussed in detail in the next section.

A Short Table of One-Sided Transform Pairs	
$x(t),\ t \ge 0^-$	$X(s)$
e^{at}, real a	$\frac{1}{s-a}$, $\mathrm{Re}[s] > a$
e^{pt}, p complex	$\frac{1}{s-p}$, $\mathrm{Re}[s] > \mathrm{Re}[p]$
$\sin(w_o t)$	$\frac{\omega_o}{s^2+\omega_o^2}$, $\mathrm{Re}[s] > 0$
$\cos(w_o t)$	$\frac{s}{s^2+\omega_o^2}$, $\mathrm{Re}[s] > 0$
$\delta(t)$, unit impulse	1
$u(t)$, unit step	$\frac{1}{s}$
$r(t)$, unit ramp	$\frac{1}{s^2}$

Questions:

1. Use the definition of the transform to verify the first entry in the table above.

2. What is the Laplace transform of the function

$$x(t) = te^{-t}$$

including the condition on $\mathrm{Re}[s]$?

3. The Laplace Transform on the Complex Plane

3.1 Its Origins

A well-behaved function can be expanded on an interval [0,T] as a Fourier series – a sum of sines and cosines with frequencies $w_k = k2p\,/\,T$ or, equivalently, as a sum of complex exponentials – as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \quad e^{jw_k t} \quad , \ 0 \le t < T$$

where

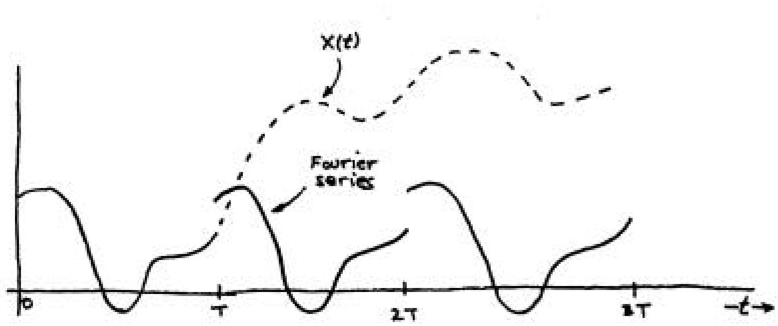
$$X_k = \frac{1}{T} \int_0^T x(t) \; e^{-j\omega_k t} dt \;\; , \omega_k = k 2\pi / T$$

since

$$\frac{1}{T} \int_0^T x(t) \; e^{-j\omega_k t} dt \; = \sum_{n=-\infty}^{\infty} \int_0^T X_n e^{j\omega_n t} e^{-j\omega_k t} dt = T \quad X_k$$

The series is a resolution of $x(t)$ into components at frequencies ω_k that are spaced by $2\pi/T$ rad/s, each component having complex amplitude X_k.

The resulting series represents the function on [0,T], although it may depart from the function outside that interval, as shown in the sketch.



Now let us increase T, the length of the interval over which the series equals the original function. The frequencies ω_k will become more closely spaced, and in the limit the sum becomes an integral

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \; e^{j\omega t} \, d\omega$$

where

$$X(\omega) = \int_{0^-}^{\infty} x(t) \; e^{-j\omega t} \; dt$$

This type of resolution into components along a continuous frequency axis is known as the Fourier transform, and the pair of integrals is known as the Fourier transform pair. The component of $x(t)$ at frequency ω, $X(\omega)$, can be considered a density: if the units of $x(t)$ are volts, then the units of $X(\omega)$ are volt-sec (or volt/Hz if we had been using Hz instead of angular frequency in radian/sec). Equivalently, the units of $X(\omega)\,d\omega$ are volts. The transform $X(\omega)$ is complex, to represent both magnitude and phase.

When we attempt to calculate the Fourier tranform of some functions, however, the integral does not converge. Examples are the unit step $x(t) = u(t)$, or $\sin(w_o t)$ or $\exp(at)$ for a>0 (try them and see). A way around this difficulty is first to multiply $x(t)$ by $e^{-s\,t}$, for some real $s > 0$, and Fourier transform the result. With s large enough, most useful functions will converge. We can easily regain $x(t)$ by inverting the modified Fourier transform and multiplying the result by $e^{s\,t}$. The pair of equations now becomes

$$X(w) = \int_{0^-}^{\infty} x(t) \quad e^{-(s + jw)\,t} \quad dt$$

$$x(t) = \frac{1}{2p} \int_{-\infty}^{\infty} X(w) \quad e^{(s + jw)\,t} \quad dw$$

Finally, a change of variable gives the Laplace transform. Define the complex frequency variable $s = s + jw$. Then

$$X(s) = \int_{0^-}^{\infty} x(t) \quad e^{-s\,t} \quad dt$$

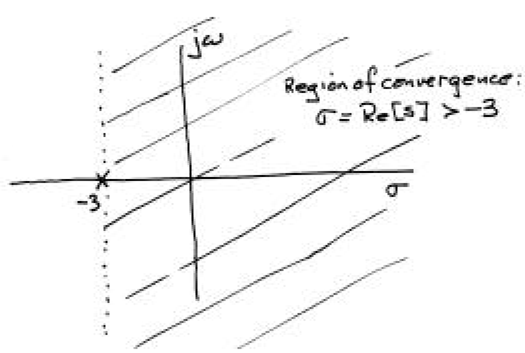
$$x(t) = \frac{1}{2p\,j} \int_{c-j\infty}^{c+j\infty} X(s) \quad e^{s\,t} \quad ds$$

where we pick $c = s$ to be large enough for convergence of the transform.

The values of s for which the real part s is large enough make up the “region of convergence”. Consider, for example, transforming $x(t) = \exp(-3t)$:

$$X(s) = \int_{0^-}^{\infty} e^{-3t} \ e^{-st} dt = \int_{0^-}^{\infty} e^{-(3+s)t} \, dt = \frac{1}{s+3}, \quad \mathrm{Re}[s] > -3$$

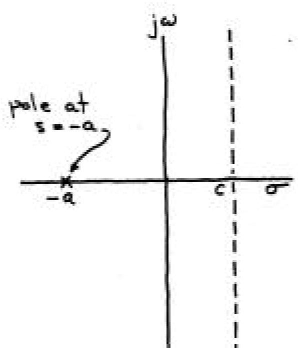
The pole and region of convergence are illustrated below:

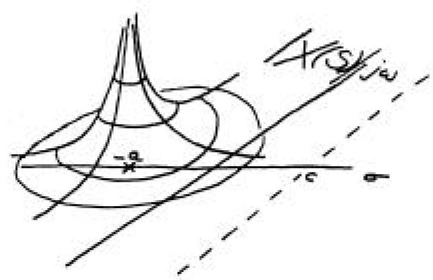


3.2 Inversion by Line and Contour Integrals

A Laplace transform $X(s)$ is apparently only going to be evaluated at points along the line $s = c + jw$ in the inverse transform integral. Nevertheless, $X(s)$ is a well-defined complex function everywhere in the complex plane $s = s + jw$ (except at the poles, which are singularities).

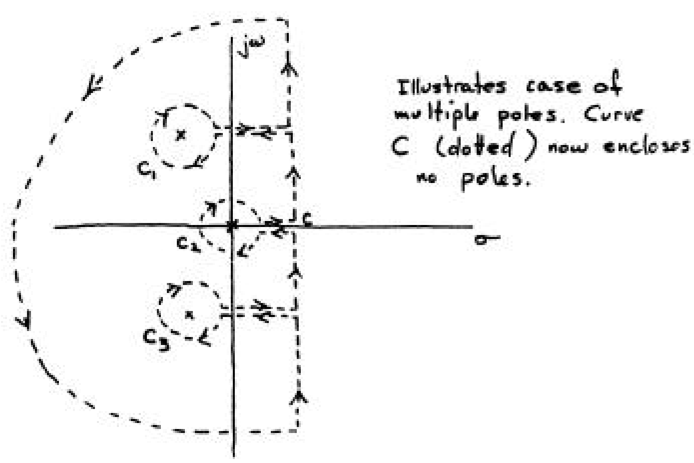
Consider the transform pair $x(t) = e^{-a t}$ and $X(s) = 1/(s+a)$. The two sketches show clearly that the inverse transform is a line integral on the complex plane.





The region of convergence consists of all points to the right of s=-a – and generally, for one-sided tranforms, it is to the right of the rightmost pole – and the integration line must lie in the region of convergence. That is, c>-a.

The line integral suggests a contour integration. Consider the integration path shown below. Curve C consists of the original line integral, circles around the poles, and a giant semicircle to the left. The cuts do not have to be considered, since the integration in one direction is cancelled by the contribution from the opposite direction, which has opposite sign.



Now for some complex variable theory. First, the Cauchy Integral Theorem: if a function $F(s)$ is analytic within and on a closed curve C then

$$\oint_C F(s) \quad ds = 0$$

Rational polynomial transforms are analytic everywhere, except at the poles. The practical result, demonstrated by the sketch above, is that our line integral equals the sum of integrals around each of the poles, less the integral along the semicircle.

Next consider the integration along the semicircle. If the integrand in

$$x(t) = \frac{1}{2p\ j} \int X(s) \quad e^{s\,t} \quad ds$$

approaches zero more quickly than $1/|s|$ on that semicircle, then the contribution from the semicircle is negligible, and approaches zero as the radius increases. This condition is ensured by the combination of (1) positive t and large negative s in the exponential factor and (2) $|X(s)| \to 0$ uniformly as the radius becomes large. The latter condition is in turn guaranteed if the degree of the numerator of $X(s)$ is less than the degree of the denominator.

We now have the important result that our line integral equals the sum of integrals around each of the poles, so that

$$x(t) = \sum_i \oint_{C_i} \frac{1}{2p\ j} \quad X(s) \quad e^{s\,t} \quad ds$$

where C_i is a closed curve encircling the i^{th} pole. Next, we use the Cauchy Integral Formula: if $F(s)$ is analytic within and on a closed curve C_i, and the point s=a is within C_i, then

$$\frac{1}{2p\ j} \oint_{C_i} \frac{F(s)}{s-a} \quad ds = F(a)$$

$F(a)$ is the "residue" of the integrand $F(s)/(s\text{-}a)$ when it is integrated around the curve centred on s=a. This is an extremely useful result. For example, we might want the inverse transform $x(t)$ of $X(s) = \dfrac{1}{s+r}$. Then, as shown by the sketch above, we must integrate $e^{s\,t}/(s+r)$ along a closed curve about the pole s=-r. In order to use the Cauchy Integral Formula, we identify $F(s)$ with e^{st} and a with $-r$. The inverse $x(t)$ is obtained in one step: $x(t) = e^{-r\,t}$, the residue at s=-r, which agrees with the result obtained in Section 1.

To obtain the residue at a pole with multiplicity n greater than one, note that differentiation of the Cauchy Integral Formula with respect to a , n-1 times, yields

$$\frac{1}{(n-1)!} \left. \frac{d^{n-1}F(s)}{ds^{n-1}} \right|_{s=a} = \frac{1}{2\pi j} \oint_{C_i} \frac{F(s)}{(s-a)^n} \; ds$$

which again lets us evaluate the residue on the right hand side without explicitly performing the integration (although the differentiation that replaces it can become tedious for integrands with many factors or for pole multiplicities greater than about three).

4. Inversion by Residues

In this section, we'll turn the contour integration and residue theory into a relatively mechanical procedure for inverting a Laplace transform of the rational polynomial type.

Given a Laplace transform $X(s)$, we want the associated inverse transform $x(t)$. The integrand in the inverse transform equation is then $X(s)e^{st}$. Define a *pole* of the integrand as a singularity point at which it becomes infinite (a loose definition, but it works for polynomials). For our rational polynomials, poles are roots of the denominator. For example, the integrand

$$\frac{\left(s^2+1\right) \; e^{st}}{\left(s-2\right)\left(s+3\right)^2\left(s^2+2s+5\right)}$$

has three simple poles, at s=2 and $s=-1\pm j2$, and a double pole at s=-3. Let there be N distinct poles p_i, $i=1,\ldots,N$, each of multiplicity n_i . In our example, $N=4: p_1=2$, $p_2=-3$, $p_3=-1+j2$, $p_4=-1-j2$ and $n_1=1$, $n_2=2$, $n_3=1$ and $n_4=1$. Each pole has an associated residue, and the inverse transform is just the sum of the all the residues. In the discussion below, we'll see how to calculate the residue at each pole.

First, define the *pole coefficient* associated with pole p_i as

$$F_i(s) = \left(s-p_i\right)^{n_i} X(s) \; e^{st}$$

In our example, the pole coefficient at $p_2=-3$ is

$$F_2(s) = \frac{\left(s^2+1\right) \; e^{st}}{\left(s-2\right)\left(s^2+2s+5\right)}$$

Notice that this definition simply removes the pole in question.

Next, we operate on the pole coefficients to produce the residue at pole p_i. For a simple pole the residue $R_i(t)$ is

$$R_i(t) = F_i(s_i)$$

In our example,

$$R_1(t) = \frac{5e^{2t}}{25 \cdot 13} = \frac{1}{65}e^{2t}$$

For a multiple pole, the residue is given by

$$R_i(t) = \frac{1}{(n_i - 1)!} \left. \frac{d^{n_i - 1} F_i(s)}{ds^{n_i - 1}} \right|_{s = p_i}$$

which requires calculation of the $(n_i - 1)^{th}$ derivative. This definition is clearly valid for simple poles, too. In our example, the residue at $p_2 = -3$ is

$$R_2(t) = -\frac{1}{4}\left(\frac{1}{10} + t\right)e^{-3t}$$

after differentiation, substitution and simplification.

The complex poles require some comment. Although they are simple poles, and do not need differentiation, the resulting arithmetic is complex. That's the bad news. The good news will be evident shortly, but first we obtain the residue at $p_3 = -1 + j2$. After factoring the quadratic in the denominator to $(s - p_3)(s - p_3^*)$, or $(s + 1 - j2)(s + 1 + j2)$, we have the residue

$$R_3(t) = F_3(p_3) = \frac{\left((-1 + j2)^2 + 1\right)e^{(-1 + j2)t}}{(-1 + j2 - 2)(-1 + j2 + 3)^2(-1 + j2 + 1 + j2)}$$

$$= \frac{(-2 - j4)e^{-t}e^{j2t}}{96 - j64} = \left(\frac{1}{208} - j\frac{1}{26}\right)e^{-t}e^{j2t}$$

The complex arithmetic was laborious, and we still have another complex pole ($p_4 = p_3^*$) to go. Now for the good news. It is straightforward to show that the residues of complex conjugate poles are themselves complex conjugates, so that $R_4(t) = R_3^*(t)$. When we sum the residues, their sum just produces $2\mathrm{Re}\left[R_3(t)\right]$, so there is no need to calculate $R_4(t)$.

The inverse transform is, at last, just the sum of the residues:

$$x(t) = \sum_{i=1}^{N} R_i(t)$$

In our example,

$$\begin{aligned} x(t) &= \frac{1}{65}e^{2t} - \frac{1}{2}\left(\frac{1}{10}+t\right)\ e^{-3t} + e^{-t}2\mathrm{Re}\left[\left(\frac{1}{208} - j\frac{1}{26}\right)e^{j2t}\right] \\ &= \frac{1}{65}e^{2t} - \frac{1}{2}\left(\frac{1}{10}+t\right)\ e^{-3t} + \frac{e^{-t}}{104}\left(\cos(2t) + 8\sin(t)\right) \end{aligned}$$

where the bracketed quantity in the last term can also be written as $\sqrt{65}\cos\left(2t - \tan^{-1}(8)\right)$.

Questions

3. Identify the poles and zeroes and their multiplicity in the transforms below:

(a) $X(s) = \dfrac{s+3}{s^3 + s^2 + 8s - 10}$

(b) $X(s) = \dfrac{s^2}{s^3 + 7s^2 + 15s + 9}$

and show their locations on the s-plane.

4. Invert the following transforms:

(a) $X(s) = \dfrac{2}{s+3}$

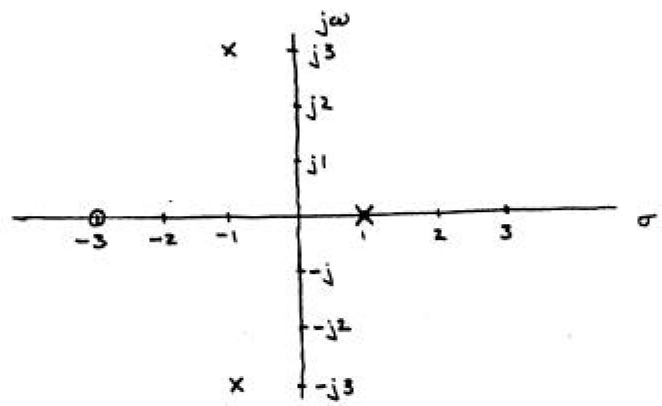
(b) $X(s) = \dfrac{s}{s^2 - 5s + 6}$

(c) $X(s) = \dfrac{1}{s^2 + 2s + 5}$

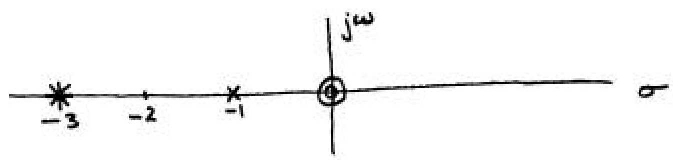
(d) $X(s) = \dfrac{s}{s^2 + 8s + 16}$

Answers

3. (a) Poles at $s=-1\pm j3$, zero at $s=-3$, all simple.



(b) Poles at $s=1$, $s=-3$ (double), zero at $s=0$ (double)



4. (a) $x(t)=2e^{-3t}$, $t\geq 0$ single pole

(b) $x(t)=3e^{3t}-2e^{2t}$, $t\geq 0$ two simple poles

(c) $x(t)=\frac{1}{2}e^{-t}\sin(2t)$, $t\geq 0$ complex conjugate poles

(d) $x(t)=(1-4t)\ e^{-4t}$, $t\geq 0$ double pole