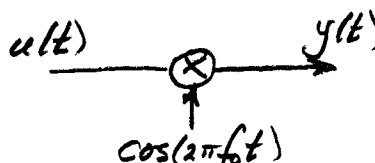


SIMON FRASER UNIVERSITY
School of Engineering Science

Solutions to Midterm Exam 97-1

1 a) $\int_{-\infty}^{\infty} 8 \sin\left(\frac{\pi}{2}t\right) \delta(t-3) dt = 8 \sin\left(\frac{\pi}{2} \cdot 3\right) = -8$ (also see p. 4)

b) The derivative property does not hold for LTV systems. First, a counterexample:



$$y(t) = \dot{u}(t) \cos(2\pi f_0 t) - 2\pi f_0 u(t) \sin(2\pi f_0 t)$$

$$\neq \dot{u}(t) \cos(2\pi f_0 t) \quad (\text{the response to deriv of input})$$

More generally, the usual convolution relation for LTI:

$$y(t) = \int u(t-\alpha) h(\alpha) d\alpha$$

becomes

$$y(t) = \int u(t-\alpha) h(t, \alpha) d\alpha$$

for a system in which the impulse response varies with time. In this case the derivative of the output is

$$\text{LTV: } \dot{y}(t) = \int \left(\dot{u}(t-\alpha) h(t, \alpha) + u(t-\alpha) \frac{\partial h(t, \alpha)}{\partial t} \right) d\alpha$$

$$\text{unlike LTI: } \dot{y}(t) = \int \dot{u}(t-\alpha) h(\alpha) d\alpha$$

2. Impulse response of the combined system is $h(k) = h_1(k) \oplus h_2(k) = h_2(k) \oplus h_1(k)$, so order of connection is immaterial. By explicit convolution, or by this summation;

k :	0	1	2	3	4	5	
$5h_1(k)$:	5	10	15	0	0	0	
$-2h_1(k-1)$:		-2	-4	-6	0	0	
$h(k)$:	5	8	11	-6	0	0	0 ← composite system

3. a) The square wave is periodic, so has a discrete spectrum, with tones at multiples of the fundamental (odd multiples for the square wave). Multiplication by $H(f)$ selects the $n = \pm 5$ harmonics at ± 10 kHz.

What is the amplitude of the harmonics? From the notes, page 2.8.4, use of $P = 2A = 2$ volt gives the $n = 5$ term

$$x(t) = \frac{4}{5\pi} \sin(2\pi 10^4 t) \text{ volts}$$

also see p(5)

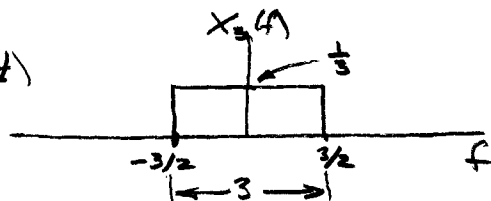
- b) The output of a system driven by a periodic input is also periodic, (To be more precise, the system must be LTI; it's not true in general for LTV).

The reason: A periodic signal has a spectrum of impulses (in frequency) at multiples of the fundamental freq. It retains this character, even after multiplication by the frequency response of the system.

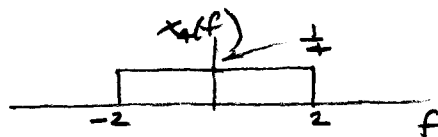
also see p(6)

4. The easy way? Well, we certainly don't want to do an explicit convolution. In the frequency domain, it's a product.

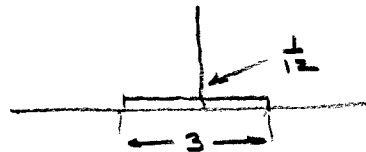
$$x_3(t) = \text{sinc}(3t) \longleftrightarrow X_3(f) = \frac{1}{3} \text{rect}_3(f)$$



$$x_4(t) = \text{sinc}(4t) \longleftrightarrow X_4(f) = \frac{1}{4} \text{rect}_4(f)$$



$$\text{So } X(f) = X_3(f)X_4(f) = \frac{1}{12} \text{rect}_3(f)$$

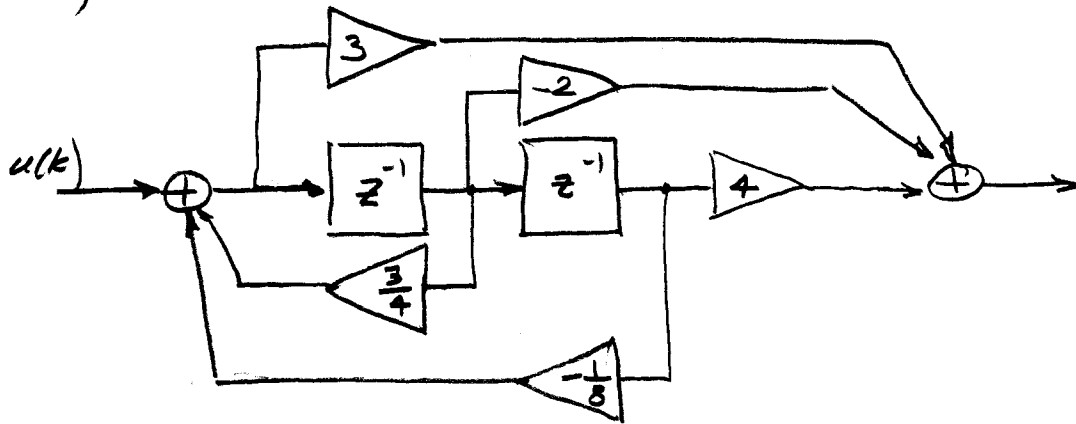


and the inverse transform is

$$x(t) = \frac{1}{4} \text{sinc}(3t)$$

This is a strange property of sinc functions.

5. d)



b) Stability. Check roots of homogeneous system:

$$z^2 - \frac{3}{4}z + \frac{1}{8} = 0 \quad (z - \frac{1}{2})(z - \frac{1}{4}) = 0$$

Roots $\frac{1}{2}$ and $\frac{1}{4}$ are both inside the unit circle, so the system is stable.

c) For zero input response we ignore $u(k)$, and look at the homogeneous response:

$$y(k) - \frac{3}{4}y(k-1) + \frac{1}{8} = 0 \quad ; \quad y(-2)=1, \quad y(-1)=0$$

From part b) the homogeneous solution is

$$y_h(k) = A\left(\frac{1}{2}\right)^k + B\left(\frac{1}{4}\right)^k$$

We could iterate the ΔE forward to get $y_{zi}(k)$ at $k=0, 1$, then choose A, B to fit. But there's no input to cause a jump at $k=0$, so we might as well fit at $k=-2, -1$.

$$k=-2: 4A + 16B = 1$$

$$k=-1: 2A + 4B = 0$$

$$\text{Easy joint solution: } A = -\frac{1}{4} \quad B = \frac{1}{8}$$

$$\text{so } y_{zi}(k) = -\frac{1}{4}\left(\frac{1}{2}\right)^k + \frac{1}{8}\left(\frac{1}{4}\right)^k$$

6. A pulse response is a homogeneous solution (with perhaps some special values for $k=0, n-1$). A homogeneous solution is a linear combination of terms, each of the form β^k or, if repeated roots, $p(k)\beta^k$

where $p(k)$ is a polynomial and β is a root of the char eq'n.

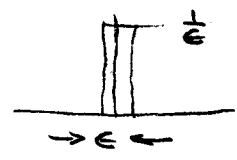
The given $h(k) = \frac{(-1)^k}{k+1}$ is not of this form, so it could not have been produced by a finite order difference equation.

Addendum to 13): Some of you obtained $\int_{-\infty}^{\infty} 8 \sin(\frac{\pi}{2}t) \delta(t-3) dt = 8 \sin(\frac{3\pi}{2})$ as I hoped, but appeared to believe that $\delta(t-3) = \begin{cases} 1 & t=3 \\ 0 & \text{otherwise} \end{cases}$

this is not correct. Don't confuse the discrete time

$$\delta(k) = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$$

with the continuous time $\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \text{rect}_{\epsilon}(t)$



which we casually think of as ∞ at $t=0$, 0 elsewhere.

one of many possible definitions

The continuous time $\delta(t)$ is "shorthand" for the limit process, giving properties wrt to integration, such as

$$\int_{-\infty}^{\infty} f(x) \delta(x-2) dx = f(2)$$

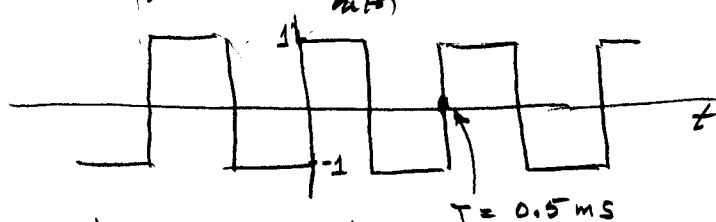
By limits, it's
$$\lim_{\epsilon \rightarrow 0} \int_{2-\epsilon/2}^{2+\epsilon/2} \frac{1}{\epsilon} f(x) dx = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F(2+\epsilon/2) - F(2-\epsilon/2)) = f(2)$$

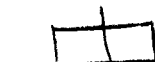
Addendum to 3a:

(5)

Here's the solution in detail:

The input is a square wave:



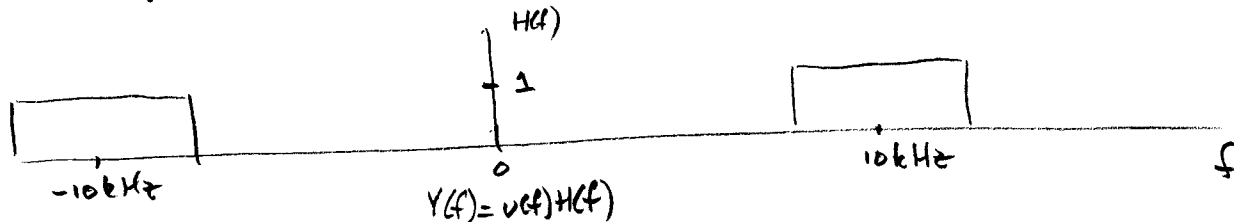
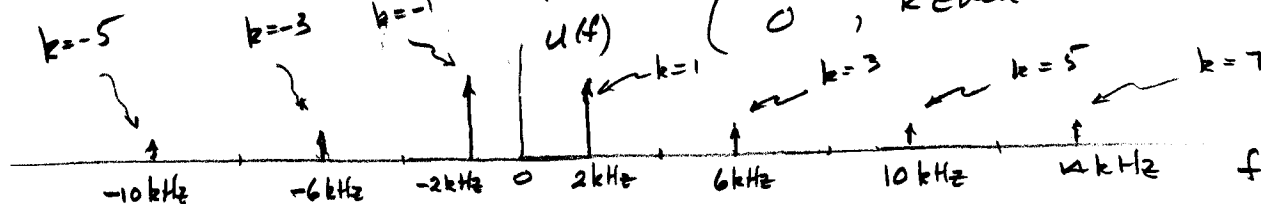
By the way, this  is not a square wave, it is a rectangular pulse.

so it can be considered as a sum of tones (sine waves) at multiples of 2 kHz, with amplitudes determined by Fourier series.

$$u(t) = \sum_k u_k e^{j2\pi k f_0 t} \quad \text{where } f_0 = \frac{1}{T} = 2 \times 10^3 \text{ Hz}$$

$$\text{and } u_k = \frac{1}{T} \int_0^T u(t) e^{-j2\pi k t/T} dt = \frac{1}{T} \int_0^{T/2} e^{-j2\pi k t/T} dt - \frac{1}{T} \int_{T/2}^T e^{-j2\pi k t/T} dt$$

$$= \frac{e^{-j\pi k} - 1}{-j\pi k} = \frac{(-1)^k - 1}{-j\pi k} = \begin{cases} \frac{2}{j\pi k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$



$$\text{We have } u(f) = \sum_k u_k \delta(f - kf_0), \quad Y(f) = u_{-5} \delta(f + 5f_0) + u_5 \delta(f - 5f_0)$$

$$\text{so } y(t) = u_{-5} e^{-j2\pi 5 f_0 t} + u_5 e^{j2\pi 5 f_0 t} \quad (\text{note } u_{-5} = u_5^*) \\ = 2 \operatorname{Re} [u_5 e^{j2\pi 5 f_0 t}] = \frac{4}{5\pi} \sin(2\pi f_0 t) = \frac{4}{5\pi} \sin(4000\pi t)$$

Addendum to 3 b)

6

We can also argue it in the time domain, of course. If we have

$$y(t) = \int h(\alpha) u(t-\alpha) d\alpha$$

Now if the input $u(t)$ is periodic with period T , then

$$u(t+T-\alpha) = u(t-\alpha)$$

and

$$y(t+T) = \int h(\alpha) u(t+T-\alpha) d\alpha = \int h(\alpha) u(t-\alpha) d\alpha = y(t)$$

So $y(t)$ is also periodic with period T .