4.7 Resonance Phenomena

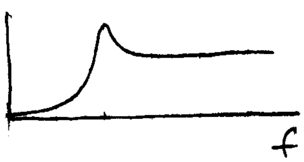
We're going to look at second order underdamped systems in a little more detail, pulling together the various threads.

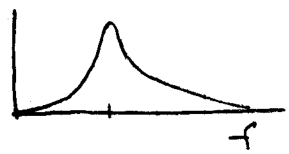
Standard Forms

It's convenient to adopt these three standard forms:

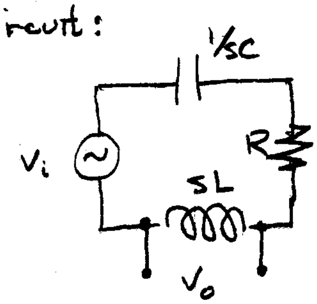
type:	hipass	bandpass	lopass
form:	$\frac{s^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$	$\frac{2\zeta\omega_0 s}{s^2+2\zeta\omega_0 s+\omega_0^2}$	$\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$
lo freq asympt:	+40 dB/dec	+20 dB/dec	1 (0 dB)
hi freq asympt:	1 (0 dB)	−20 dB/dec	−40 dB/dec

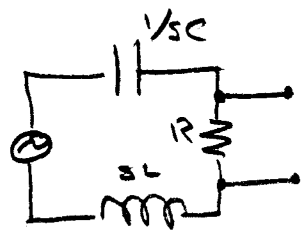
sketch:

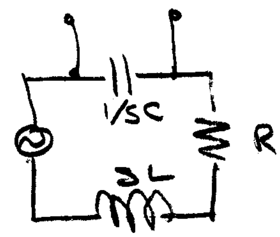




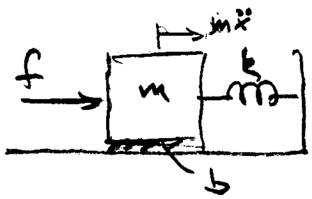
typical circuit:

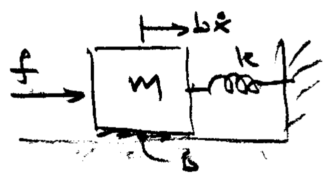


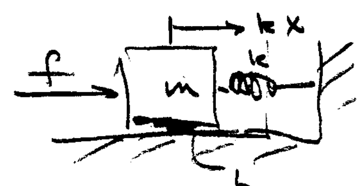




$$\frac{s^2}{s^2+\frac{R}{L}s+\frac{1}{LC}} \qquad \frac{sR/L}{s^2+\frac{R}{L}s+\frac{1}{LC}} \qquad \frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$



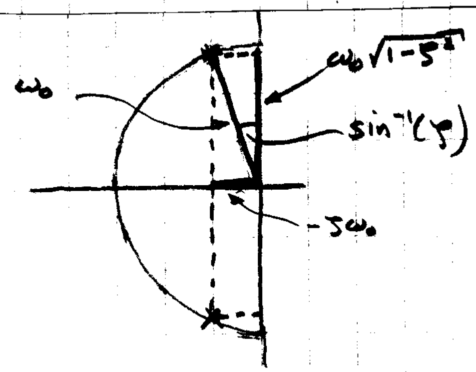




accel force | friction force | spring force

$$\frac{s^2}{s^2+\frac{b}{m}s+\frac{k}{m}} \qquad \frac{s\,b/m}{s^2+\frac{b}{m}s+\frac{k}{m}} \qquad \frac{k/m}{s^2+\frac{b}{m}s+\frac{k}{m}}$$

Pole Locations and Resonant Behaviour



pole locations $\omega_0(-\zeta \pm j\sqrt{1-\zeta^2})$, $\zeta < 1$

as ζ decreases from 1 to 0, poles move from $-\omega_0$ on real axis to $\pm j\omega_0$ on imaginary axis along a circle of radius ω_0 (natural frequency)

Now look at freq resp of lightly damped systems. Easiest to analyse the bandpass version:

$$H(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{1}{\frac{s}{2\zeta\omega_0} + 1 + \frac{\omega_0}{2\zeta s}}$$

Substitute $s = j\omega$

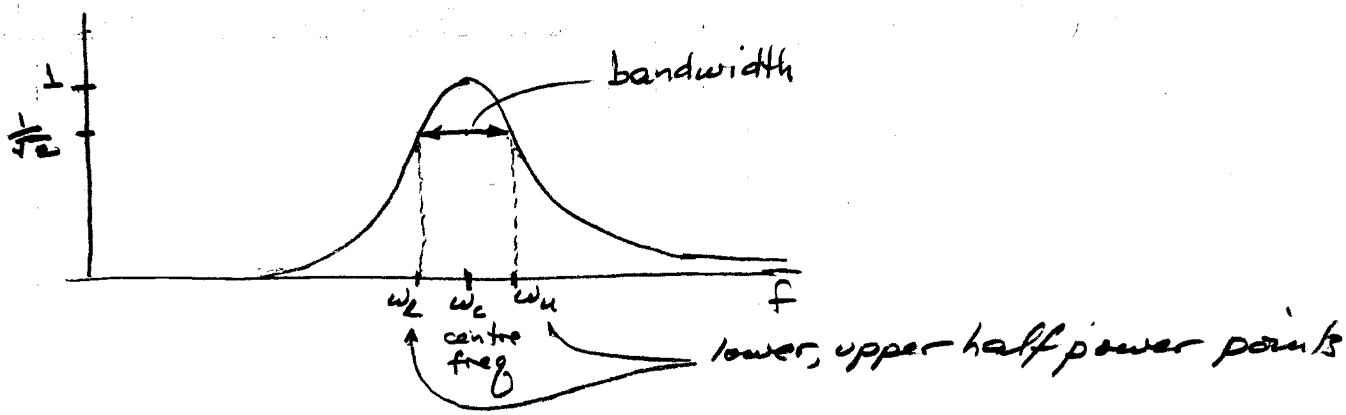
$$H(\omega) = \frac{1}{\frac{j\omega}{2\zeta\omega_0} + 1 + \frac{\omega_0}{j2\zeta\omega}} = \frac{1}{1 + j\left(\frac{u}{2\zeta} - \frac{1}{2\zeta u}\right)} \qquad u = \frac{\omega}{\omega_0}$$

$$= \frac{1}{1 + jQ\left(u - \frac{1}{u}\right)} \qquad Q = \frac{1}{2\zeta}$$

Note the geometric symmetry about ω_0: $H(\omega_0 u) = H^*(\omega_0/u)$.
This means symmetry on a log frequency scale.

We can already pick off several points on freq resp:

ω	$\|H(\omega)\|$	$\arg(H(\omega))$	comment
0	0	90°	
ω_ℓ	$\frac{1}{\sqrt{2}}$ (-3 dB)	45°	lower half pwr pt
ω_0	1 (0 dB)	0°	resonant max
ω_u	$\frac{1}{\sqrt{2}}$ (-3 dB)	-45°	upper half pwr pt, $\omega_u\omega_\ell = \omega_0^2$
∞	0	-90°	note total change of 180°



As Q increases, the 3 dB points move closer to ω_0, so the bandwidth decreases. The rapid changes in amplitude and phase take place over a smaller range of frequencies.

Although exact calculation of ω_ℓ and ω_u requires solving a quartic polynomial, a simple approx is good for high Q or ω near ω_0:

$$\text{define} \quad \epsilon = \frac{\omega}{\omega_0} - 1 = u - 1 \;; \quad \frac{1}{u} = \frac{1}{1+\epsilon} \approx 1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} \cdots$$

$$\text{and} \quad u - \frac{1}{u} \approx 1 + \epsilon - (1 - \epsilon) = 2\epsilon$$

$$\text{so} \quad H(\omega) \approx \frac{1}{1 + j\,2Q\epsilon}$$

Thus the upper & lower halfpower (3 dB) points are at $\epsilon \approx \pm 1/2Q$, or $\omega = \omega_0 \pm \omega_0/2Q$. The bandwidth:

$$\beta = \omega_u - \omega_\ell \approx \omega_0 / Q$$

So another common interpretation of Q is the reciprocal fractional bandwidth:

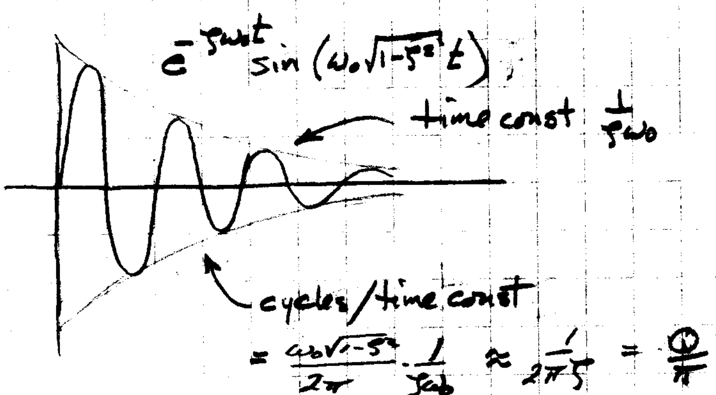
$$Q \approx \frac{\omega_0}{\beta} \qquad \text{doesn't matter if Hz or rad/s.}$$

$$\beta = \frac{\omega_0}{Q} = 2\zeta\omega_0 \,!$$

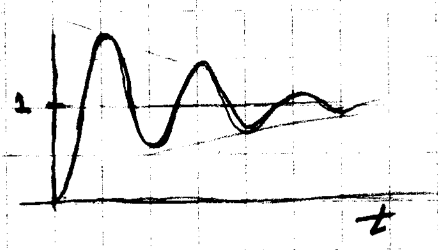
Finally, recall that we started with the bandpass form. The effect of the highpass form (no zeroes) is to shift the resonance peak up in frequency a little and to make the phase run from 180° down to 0° — to see this, just consider the effect of including another numerator factor s on the Bode plot. The lowpass form? Obvious: shifts the peak down a little, phase runs 0° to −180°. The shifts are insignificant for large Q.

Time Response and Q

lopass impulse response

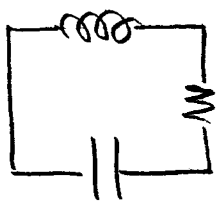


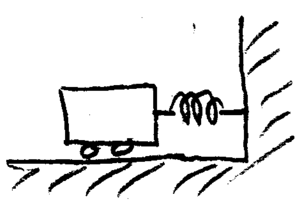
lopass step response

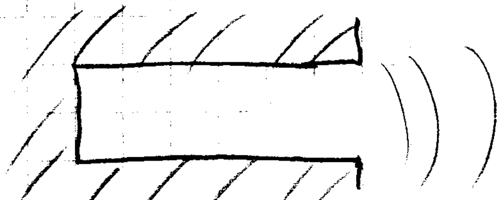


Physical Interpretation of Resonance

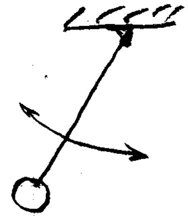
- Most resonance effects involve the interplay of two (or more) forms of energy storage.





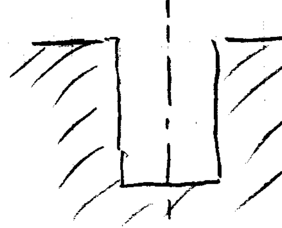


acoustic tube

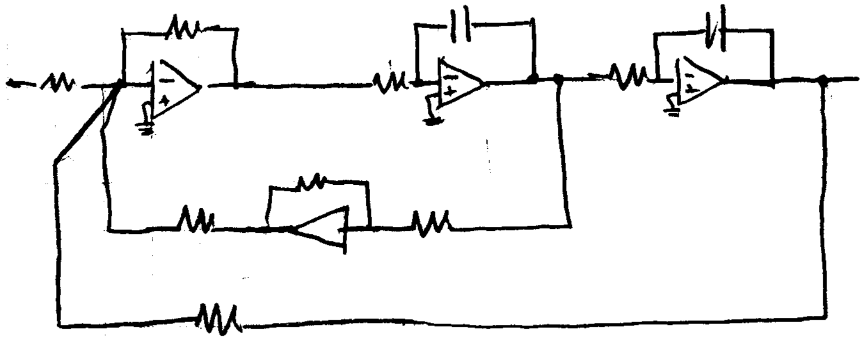


pendulum

radio

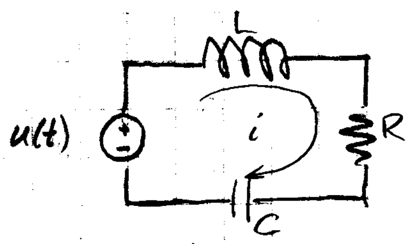


cavity resonator



although not all systems with multiple points of energy storage resonate.

- Consider RLC for more detailed discussion:



energy storage:

$$E_c(t) = \tfrac{1}{2} C v_c^2(t) \qquad E_L(t) = \tfrac{1}{2} L\, i^2(t)$$

Kick it with a unit impulse. Then observe homogeneous response

$$V_c(s) = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$v_c(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_0 t} \sin(\omega_0\sqrt{1-\zeta^2}\, t)$$

$$i(t) = C\frac{dv_c}{dt} = C\omega_0^2 e^{-\zeta\omega_0 t}\left(\cos(\omega_0\sqrt{1-\zeta^2}\, t) - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_0\sqrt{1-\zeta^2}\, t)\right)$$

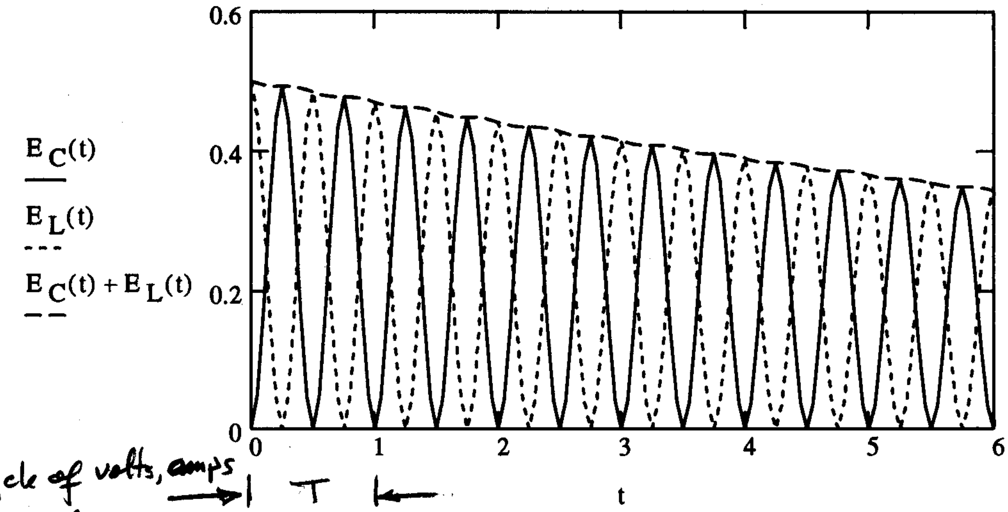
and the capacitor energy is

$$E_c(t) = \frac{1}{2} C \frac{\omega_0^2}{1-\zeta^2} e^{-2\zeta\omega_0 t} \sin^2(\omega_0\sqrt{1-\zeta^2}\, t) \approx \frac{1}{2L} e^{-2\zeta\omega_0 t} \sin^2(\omega_0 t)$$

$$E_L(t) = \frac{1}{2} L C^2\omega_0^4 e^{-2\zeta\omega_0 t}\left(\cos(\omega_0\sqrt{1-\zeta^2}\, t) - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_0\sqrt{1-\zeta^2}\, t)\right)^2$$

$$\approx \frac{1}{2L} e^{-2\zeta\omega_0 t} \cos^2(\omega_0 t)$$

$\omega_0 = 6.283 \qquad \zeta = 5\cdot 10^{-3}$



one cycle of volts, amps $\rightarrow| \quad T \quad |\leftarrow$

two cycles of energy $\qquad T = \dfrac{2\pi}{\omega_0\sqrt{1-\zeta^2}}$

During T, the decay is

$$e^{-2\zeta\omega_0 T} = \exp\left(-\frac{4\pi\zeta}{\sqrt{1-\zeta^2}}\right) \approx e^{-4\pi\zeta} = e^{-2\pi/Q}$$

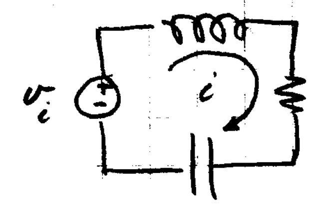
and for small ζ (high Q), $e^{-2\pi/Q} \approx 1 - \frac{2\pi}{Q}$

Therefore, the fractional energy loss per cycle is $\frac{2\pi}{Q}$.

e.g. if $Q = 628$, then system loses 1% of energy each cycle.

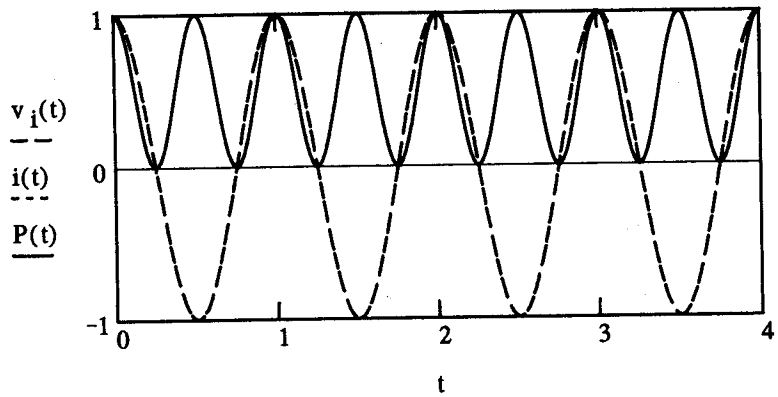
- Where does the energy go?

- If there is a source in the circuit, then its power input makes up the losses, keeping a steady amplitude out.



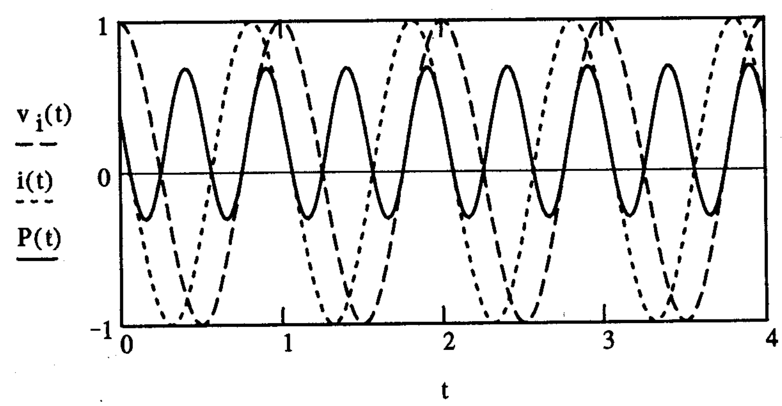
In general — not just for resonant circuits — the power flow is oscillatory, flowing out of and back into the source at different points in the cycle.

$$v_i(t) := \cos\left(2\cdot\pi\cdot f_0\cdot t\right) \quad i(t) := A\cdot\cos\left(2\cdot\pi\cdot f_0\cdot t + \phi\right) \quad P(t) := v_i(t)\cdot i(t)$$



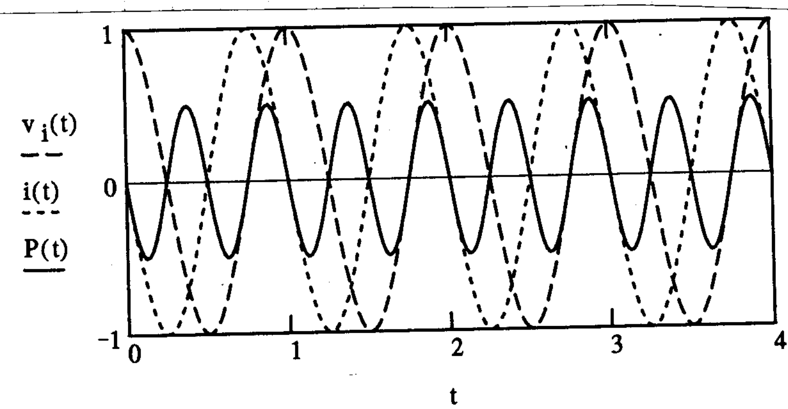
$A = 1$

$\phi = 0$



$A = 1$

$\phi = 1.178$



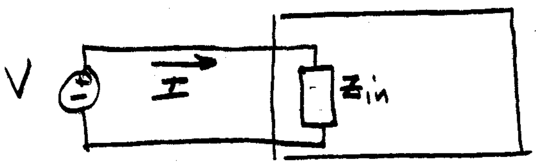
$A = 1$

$\phi = 1.571$

If $v(t) = Re\left[V e^{j2\pi f_0 t}\right]$, $i(t) = Re\left[I e^{j2\pi f_0 t}\right]$

$$P(t) = v(t)\,i(t) = \tfrac{1}{2} Re\left[V I^* + V I e^{j4\pi f_0 t}\right]$$

$$P_{av} = \tfrac{1}{2} Re\left[V I^*\right]$$



$$P_{av} = \tfrac{1}{2} Re\left[V I^*\right] = \tfrac{1}{2} Re\left[V\left(\frac{V}{Z_{in}}\right)^*\right]$$

$$= \tfrac{1}{2} |V|^2 Re\left[\frac{1}{Z_{in}}\right]$$

For our series RLC

$$Z = j\omega L + R + \frac{1}{j\omega C}$$

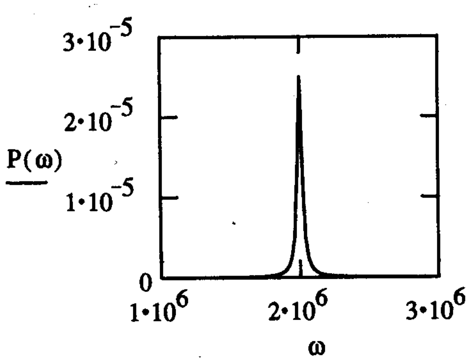
$$Z^{-1} = \frac{j\omega/L}{\omega_0^2 - \omega^2 + j2\zeta\omega_0\omega} = \frac{1}{L}\,\frac{j\omega(\omega_0^2 - \omega^2 - j2\zeta\omega_0\omega)}{(\omega_0^2-\omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}$$

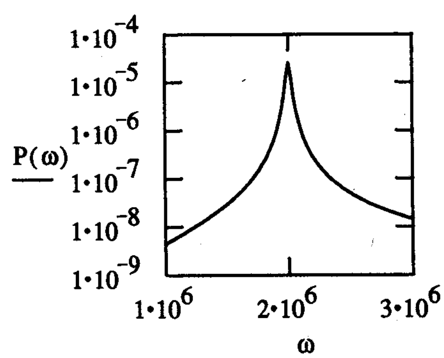
$$Re\left[Z^{-1}\right] = \frac{2\zeta\omega_0\omega^2/L}{(\omega_0^2-\omega^2)^2 + 4\zeta^2\omega_0^2\omega^2} \longrightarrow \frac{1}{R} \text{ at } \omega = \omega_0$$

$L := 0.5$ $\quad C := 10^{-6}$ $\quad \omega_0 := \dfrac{1}{L \cdot C}$ $\quad \omega_0 = 2 \cdot 10^6$

$\zeta := 0.01$ $\quad R := 2 \cdot \zeta \cdot \omega_0 \cdot L$ $\quad R = 2 \cdot 10^4$

$$P(\omega) := \frac{1}{2 \cdot L} \cdot \frac{2 \cdot \zeta \cdot \omega_0 \cdot \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(2 \cdot \zeta \cdot \omega_0 \cdot \omega\right)^2} \qquad \omega := 1 \cdot 10^6, 1.02 \cdot 10^6 .. 3 \cdot 10^6$$





Bode Plots of Resonant Systems

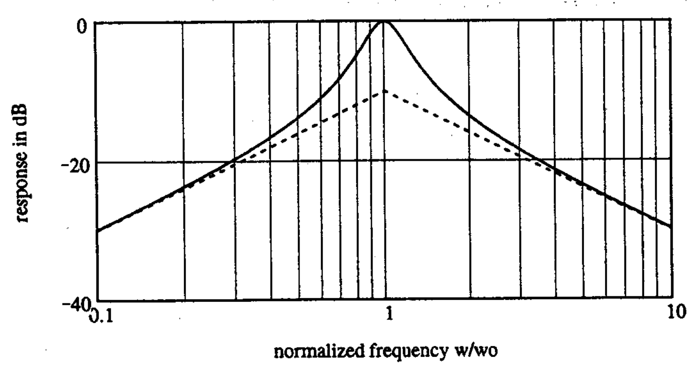
- By manipulation, we can put it in the form $H(\omega) = \dfrac{1}{1 + jQ(u - \frac{1}{u})}$

Check asymptotes:

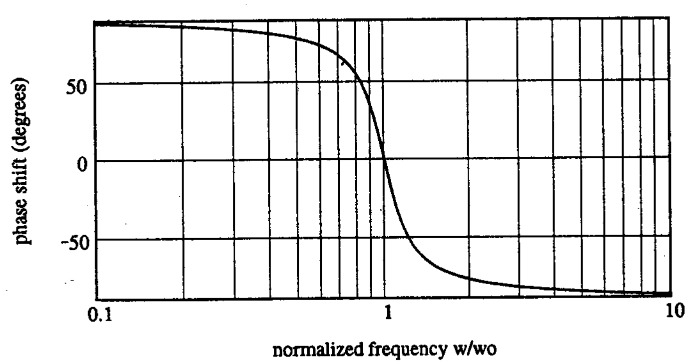
- for $\omega \to 0$ $H(\omega) \sim \dfrac{u}{jQ}$: 20 dB/dec, angle 90°

- for $\omega \to \infty$ $H(\omega) \sim \dfrac{1}{jQu}$: −20 dB/dec, angle −90°

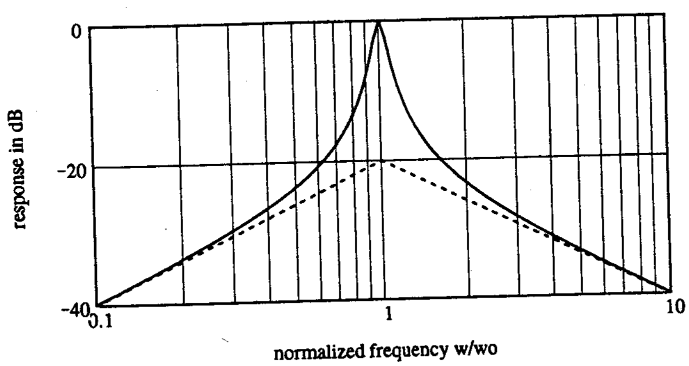
Intersection of asymptotes at $u = 1$ ($\omega = \omega_0$), where magnitude is $1/Q$. True value of $H(\omega_0) = 1$



Q = 3.16

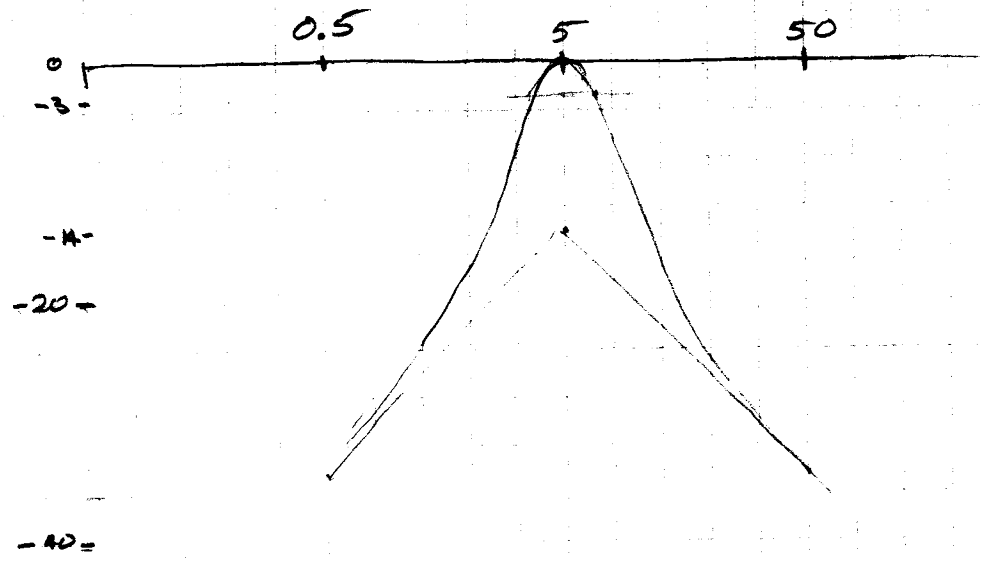


Q = 3.16

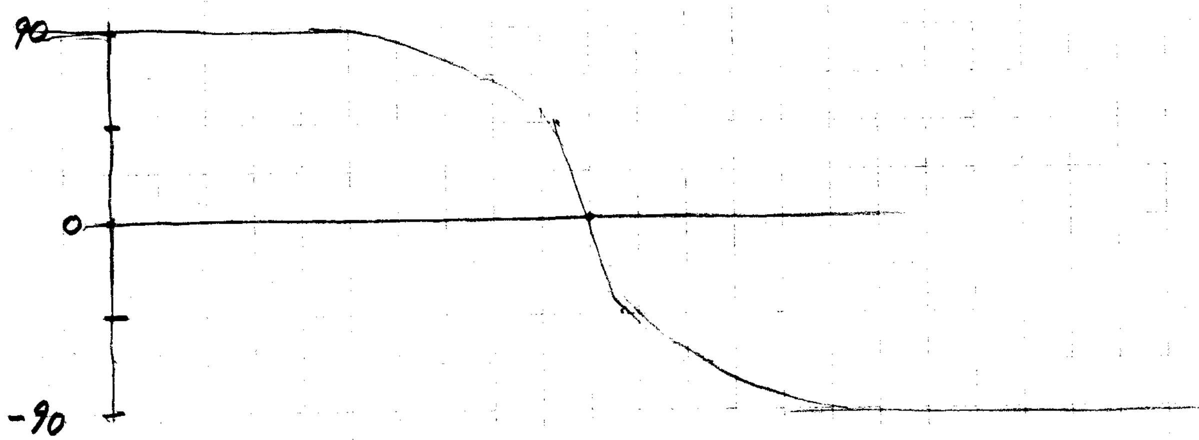


• **Example** $H(s) = \dfrac{s}{s^2 + s + 25}$ sketch Bode plot.

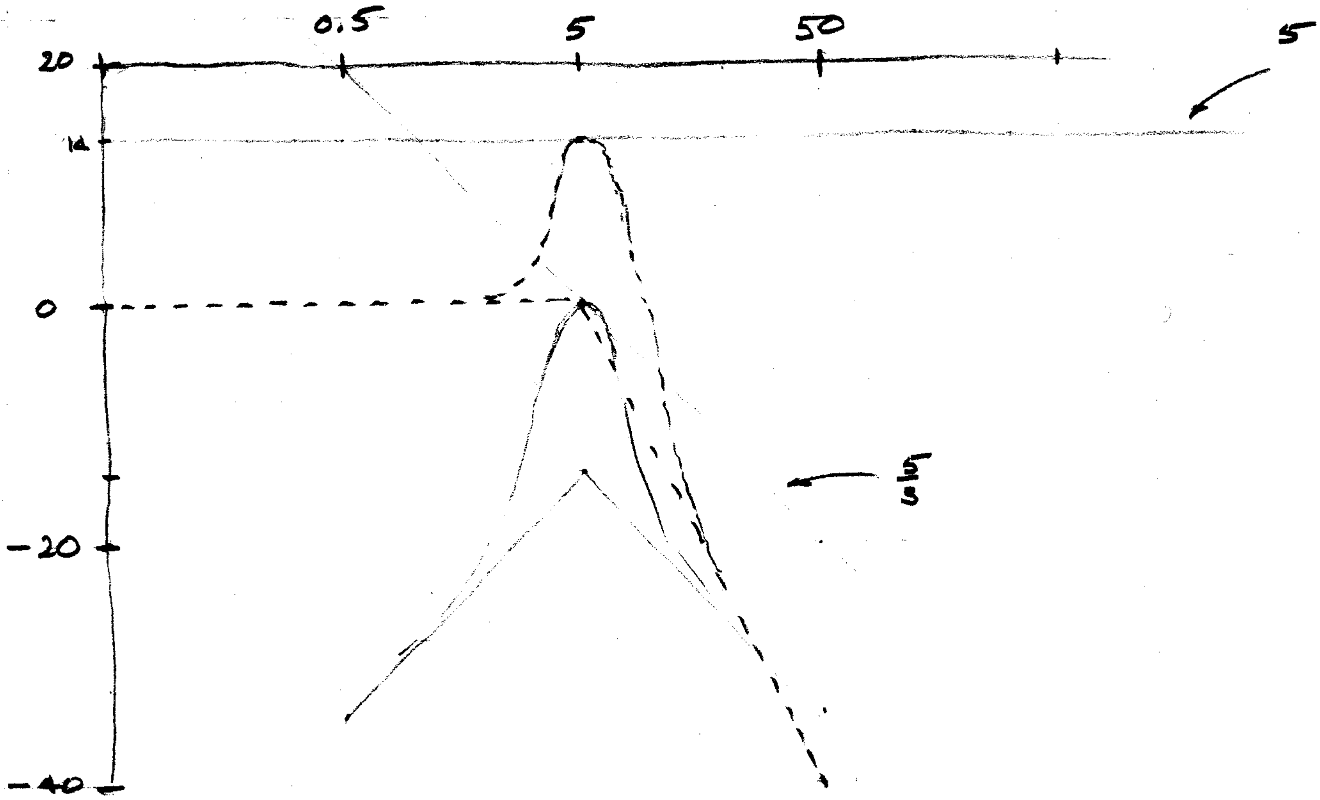
$\omega_0 = 5$, $2\zeta\omega_0 = 1$, $\zeta = 0.1$, $Q = 5$, $\beta = \frac{\omega_0}{Q} = 2\zeta\omega_0 = 1$



$20 \log Q = 14\,dB$

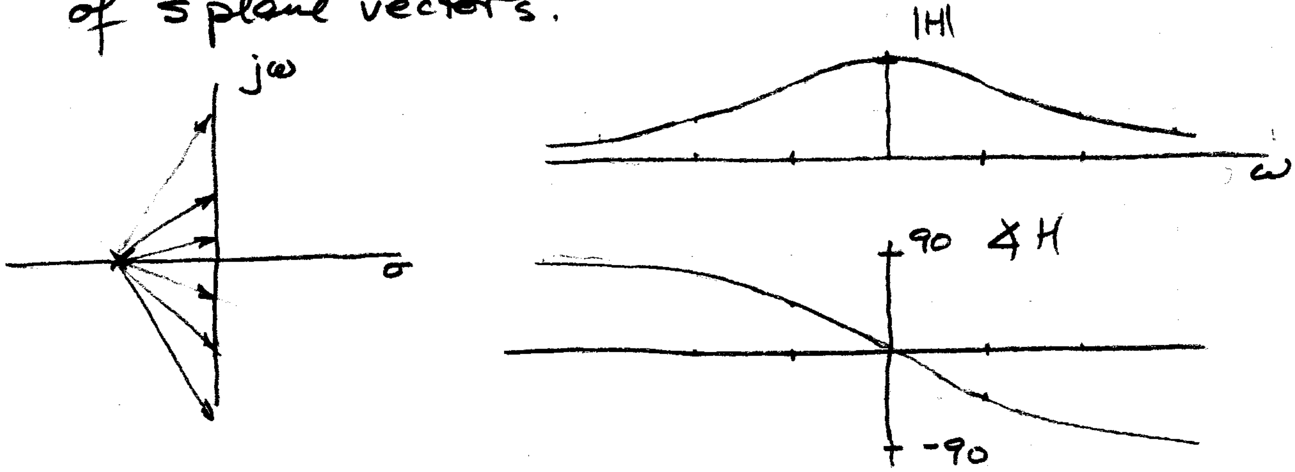


example $H(s) = \frac{25}{s^2+s+25} = \frac{25}{5}\,\frac{s}{s^2+s+25} = 5\,\frac{5}{s}\,\frac{s}{s^2+s+25}$



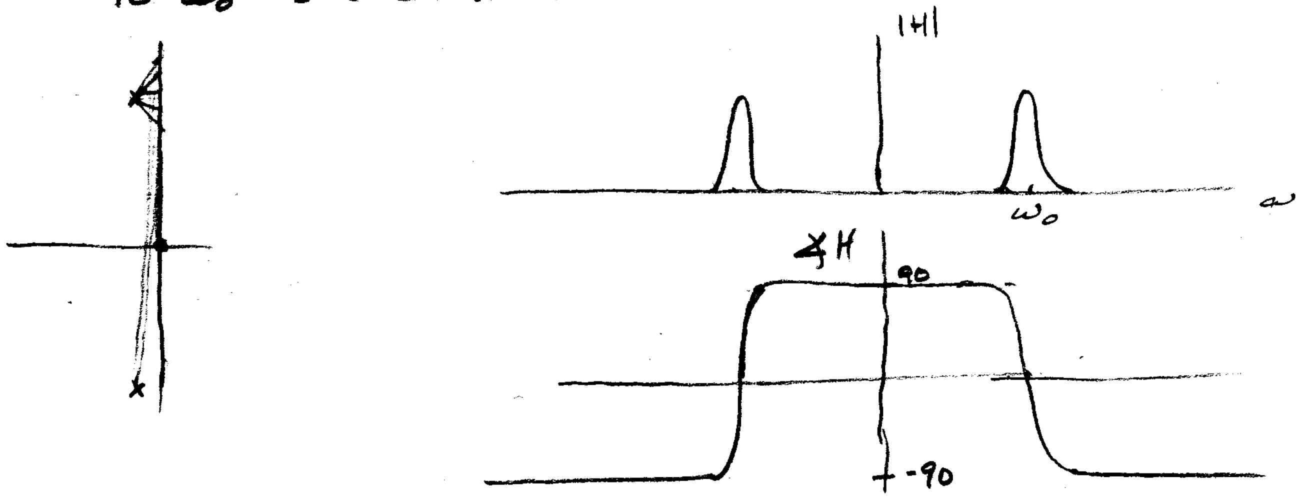
Lowpass - Bandpass Transformation

• Consider a single real pole. Frequency response for positive and negative frequencies can be sketched from consideration of s plane vectors.



This is the same behaviour as in resonant circuits, but centred at 0 Hz.

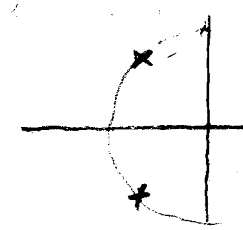
• For the bandpass resonant circuit, especially for high Q, we can see from splane vectors that its freq resp is that of the simple 1st order circuit, translated to ω_0 as a centre.

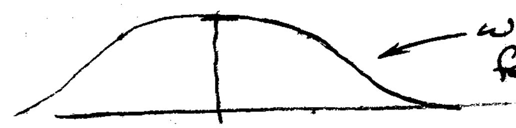


The ratio of lengths of zero vector and the lower pole vector doesn't change much as we pass resonance. The angles of these vectors approximately cancel.

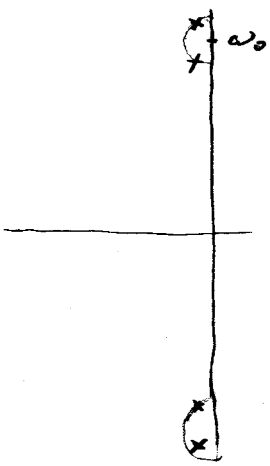
4.7.13

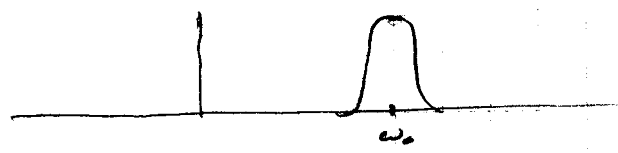
- So good lowpass filters can be converted to good bandpass filters, e.g., 2nd order Butterworth





← wider passband, faster rolloff than 1st order





This is a 4th order Butterworth BPF.

- A useful transformation: if you have a good LPF normalised to bandwidth 1 rad/s, you can obtain a bandpass by

$$H_{bp}(s) = H_{lp}\left(\frac{s^2+\omega_c^2}{s\,\omega_b}\right) \qquad \omega_c \text{ centre}; \ \omega_b \text{ bandwidth}$$

e.g. our 1st order LPF is $\frac{1}{s+1}$. To get BPF, substitute to get

$$H_{bp}(s) = \frac{1}{\frac{s^2+\omega_c^2}{s\omega_b}+1} = \frac{s\omega_b}{s^2+\omega_b s+\omega_c^2} = \frac{2\zeta\omega_0 s}{s^2+2\zeta\omega_0 s+\omega_0^2}$$

Similar transformations convert lopass to hipass or even band reject