SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 380 Linear Systems

REFRESHER ON SHIFTING, COMPRESSING AND REVERSING FUNCTIONS

1. INTRODUCTION

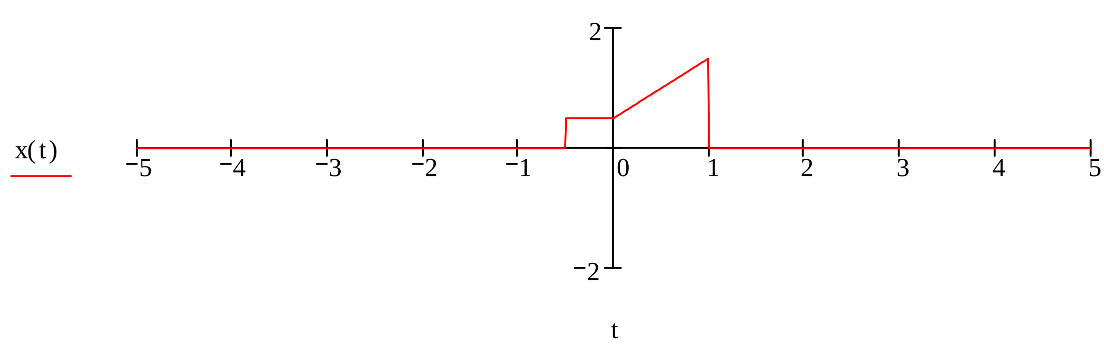
The topics of this note may seem mind-numbingly obvious - and, in fact, they are. Nevertheless, it's rustiness on these basics that causes some students to have difficulty with convolution - which is not an elementary topic and definitely not one you want to be confused about. So take 5 or 10 minutes and read through to the end. It may save you much time and puzzlement later.

We will be concerned with transformations like this: $y(t)=x(c(t-d))$, where $x(t)$ is some defined function and c and d are parameters that may be positive or negative, and sometimes even complex. We interpret the transformation as follows: to evaluate y at some t, first evaluate the argument $c(t-d)$, then evaluate the original function x at that value of argument. Of course, if we have the functional form of $x(t)$, we can simply make the substitution. For example, if $x(t)=t^2$, then $y(t)=(c(t-d))^2=c^2(t-d)^2$. But we'd also like a graphical picture of what's going on, in order to build insight - and besides, often we don't have a functional form for $x(t)$, for example, in general discussions of signals or filters.

We will use this function for all the examples below:

$$x(t) := \begin{cases} 0.5 & \text{if } (-0.5 \le t)\cdot(t<0) \\ t+0.5 & \text{if } (0 \le t)\cdot(t<1) \\ 0 & \text{otherwise} \end{cases}$$

It looks like this in the range $t := -5, -4.99 .. 5$:



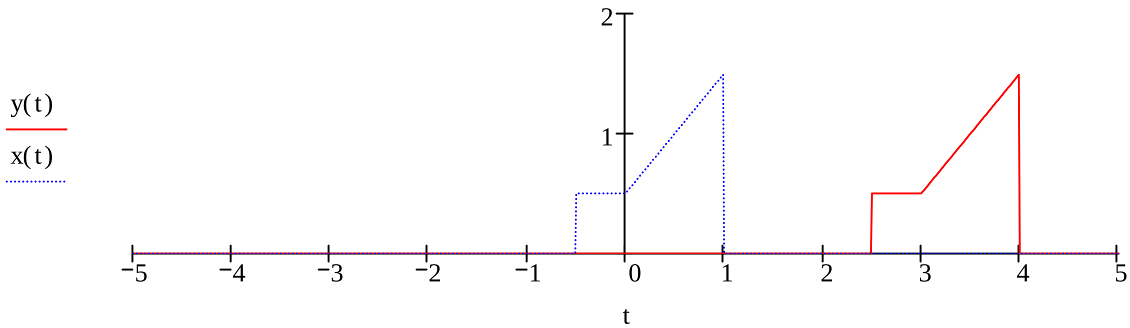
Because of its importance in the function sketching exercises below, we'll call the t=0 spot on the function the *centre*. In this case, it's where it breaks from flat to a linear rise. This is not a standard math term, nor is it really the centre of the non-zero part of the function. Still, it will be quite useful when we consider the various transformations.

2. SHIFTING A FUNCTION IN TIME

Shifting a function (which is more formally known as *translation*) is pretty straightforward. Consider

$$y(t) := x(t-3)$$

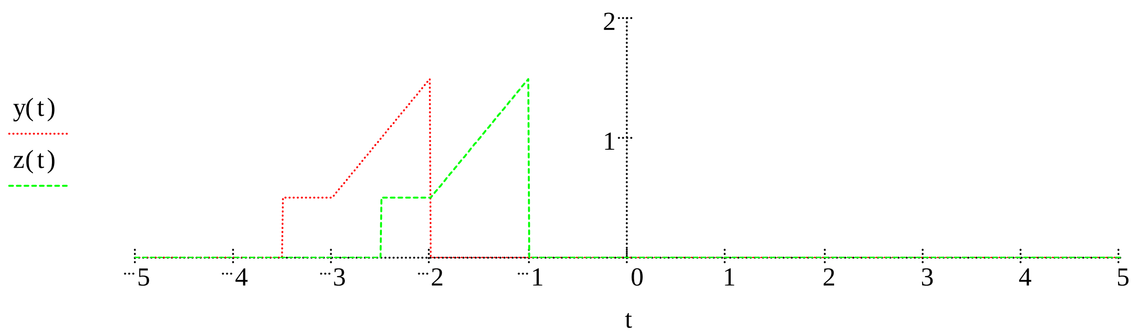
It means that, to find the value of *y* at *t*, we first evaluate the argument of *x* (subtract 3 from *t*) then find the value of *x* corresponding to this new argument. To sketch the result, simply note where the centre has gone: it corresponds to an *x* argument of zero, so *t*-3=0, hence it's at *t*=3. Check it:



The function has been delayed - shifted to the right - by 3 time units. More generally, *x*(*t*-*d*) is the function *x*(*t*) delayed (shifted to the right) by *d* units, and if *d* takes a negative value, then it's a time advance of *d* (a shift to the left), since the centre always moves to *t*=*d*. You could equally well use a different convention and write *x*(*t*+*a*) as the function *x*(*t*) advanced (shifted to the left) by *a* units, since the centre has moved to *t*=-*a*.

Below, you can choose shift values in the highlighted regions (after changing the value, press F9 or click outside the region). Try to predict where the centre will go *before* you look at the plot.

$d := -3$ $\quad y(t) := x(t-d)$ $\quad a := 2$ $\quad z(t) := x(t+a)$

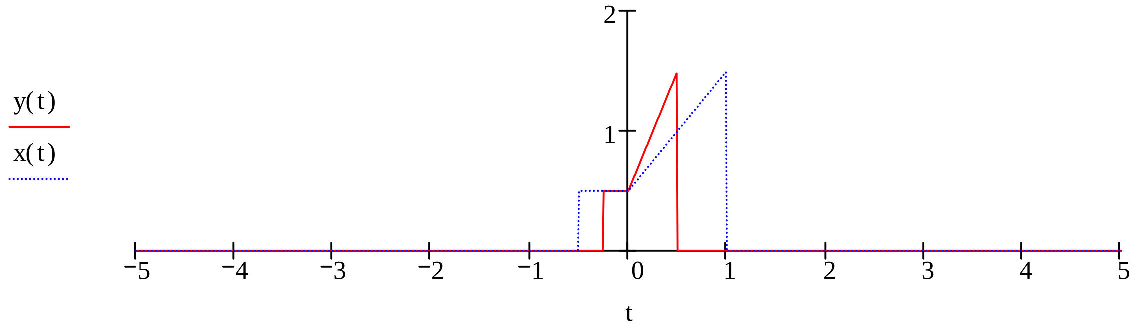


3. COMPRESSING A FUNCTION IN TIME

This operation is more formally termed *dilation* or *compression.* Consider the function:

$y(t) := x(2 \cdot t)$

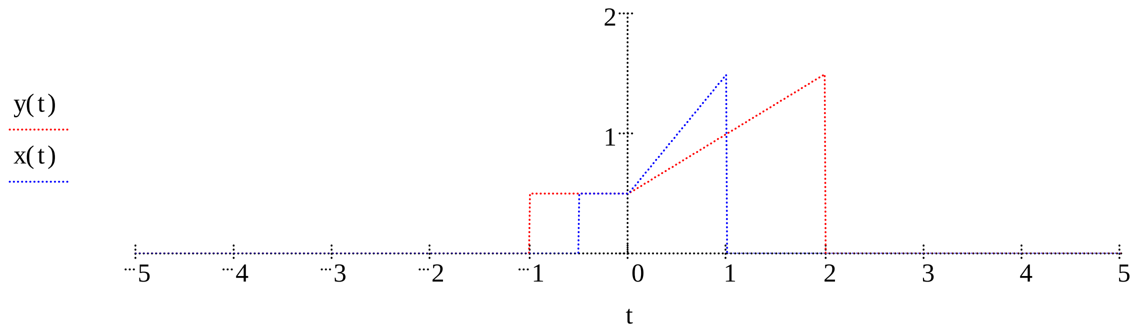
Clearly, it doesn't change the centre of the function. However, the argument of *y* changes twice as quickly as it does in *x*. Consequently, we go through time twice as quickly, which compresses the original function. Check it:



Conversely if we had multiplied *t* by 1/2, instead of 2, we would have gone through time more slowly, and the function would have been dilated.

You can try various values of *c* here . Use only positive values for the moment - we'll consider negative values in the next section

$c := 0.5$ $\qquad$ $y(t) := x(c \cdot t)$

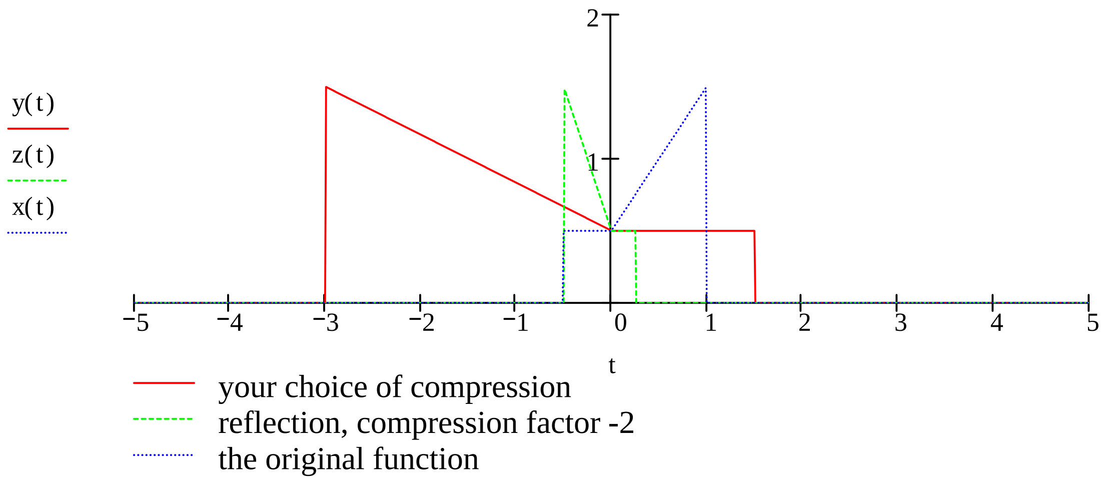


4. REVERSING A FUNCTION

The Transformation

We can generalize compression to allow negative numbers, such as *y*(*t*)=x(-*t*). Since the argument of *x* is reversed in sign, we have simply reflected the function about *t*=0, that is, through the vertical axis. The centre remains where it was. Similarly, *y*(*t*)=*x*(-2*t*) reflects and compresses the function. You can choose the parameter for the plot below:

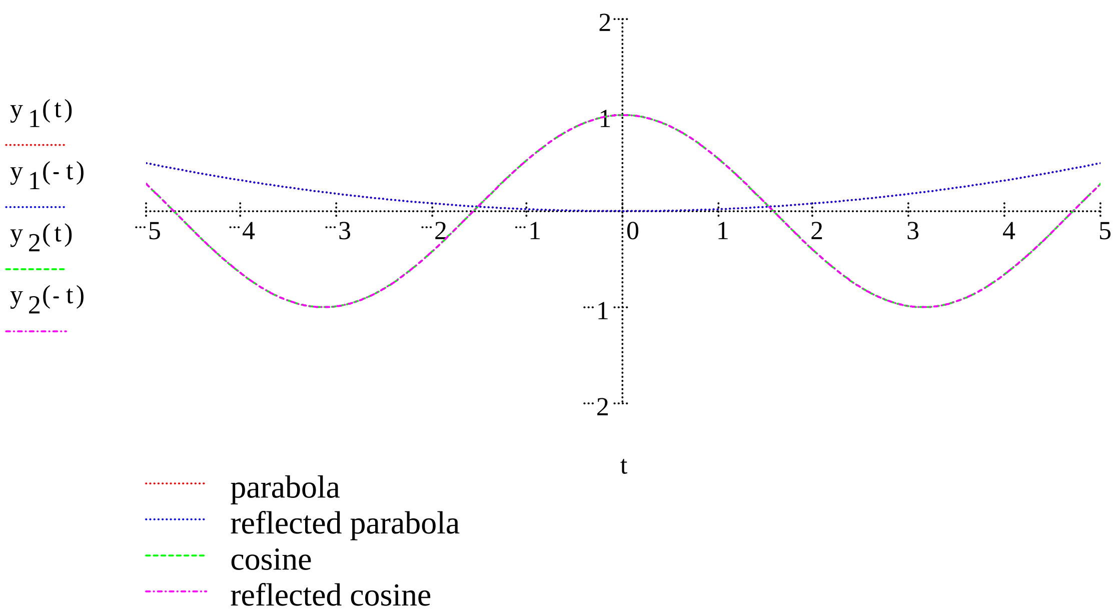
$c := \frac{-1}{3}$ $\quad y(t) := x(c \cdot t)$ $\quad z(t) := x(-2 \cdot t)$



Symmetries

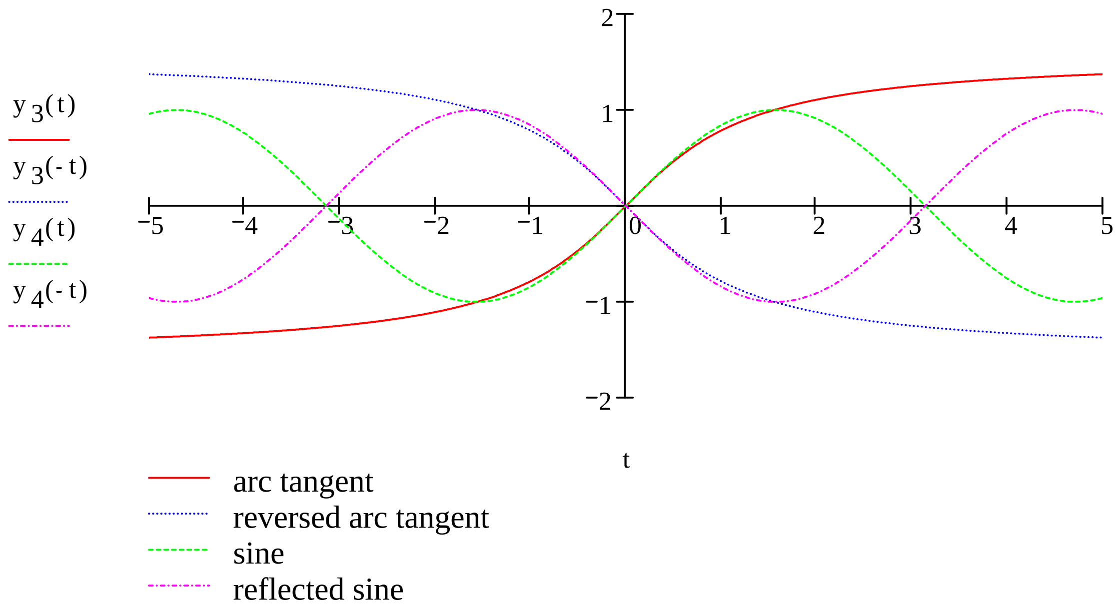
Speaking of reflections, some symmetries are important in this course. An *even function* is one that looks the same when reversed. Examples:

$y_1(t) := 0.02 \cdot t^2$ $\quad y_2(t) := \cos(t)$



Similarly, an *odd function* is one that changes its sign when reversed left to right. Examples:

$y_3(t) := \mathrm{atan}(t)$ $y_4(t) := \sin(t)$



Finally, a special symmetry for the complex functions you'll work with in this course. A function is *conjugate symmetric* if reversal causes it to be conugated. Here's an important example:

$$y_{cs}(t) := \cos(t) + j \cdot \sin(t)$$

The real part is even and the imaginary part is odd, so that

$$y_{cs}(-t) = \cos(-t) + j \cdot \sin(-t) = \cos(t) - j \cdot \sin(t) = \overline{y_{cs}(t)}$$

(overhead bar denotes conjugation in Mathcad)

This function is often called the complex exponentia $y_{cs}(t) = e^{j \cdot t}$

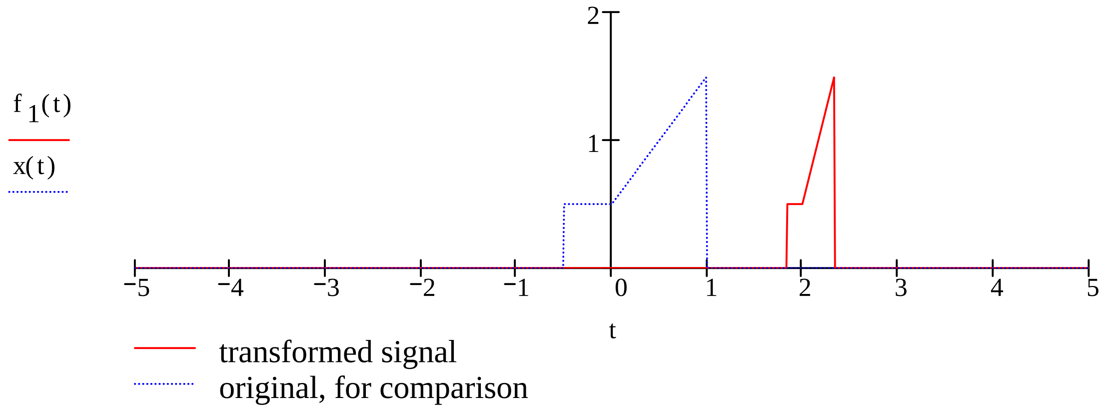
In any case, the importance of this property is that the Fourier transform (or inverse transform) of a conjugate symmetric function is real, as you'll see later in the course. Conversely, if function is real, its inverse Fourier transform (or the transform itself) is conjugate symmetric.

5. COMBINED SHIFTING, COMPRESSION AND REFLECTION

By this point, you should have no trouble with more complicated transformations. First, consider

$$f_1(t) := x(3 \cdot (t - 2))$$

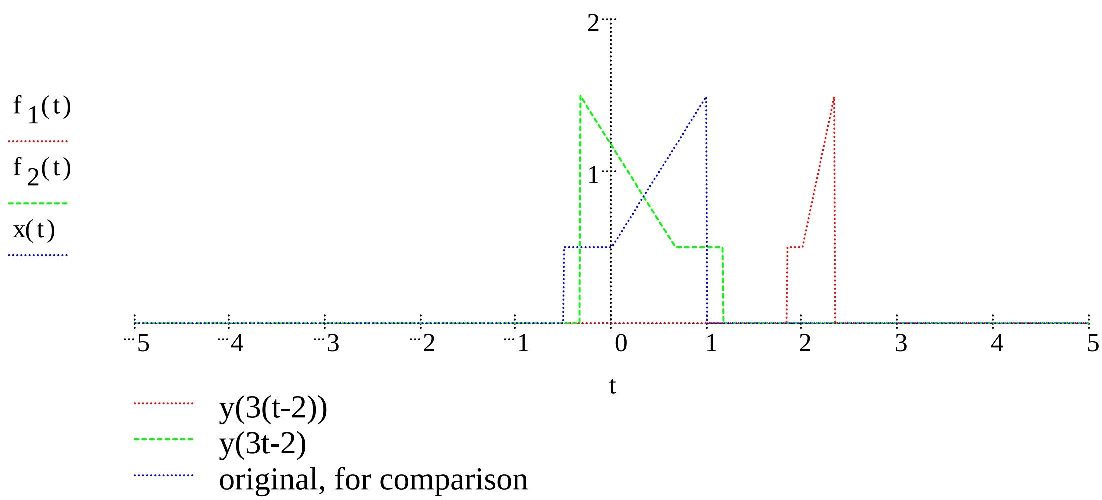
The argument of *y* is zero at *t*=2, so the centre has shifted to that spot. The domain variable *t* has a scale factor of 3, so there is compression by a factor of 3 about the new centre. Not too hard to sketch:



There is a difference between that function and this one:

$$f_2(t) := y(3 \cdot t - 2)$$ (same compression factor, but centre shifted to *t*=2/3)

Check it out:

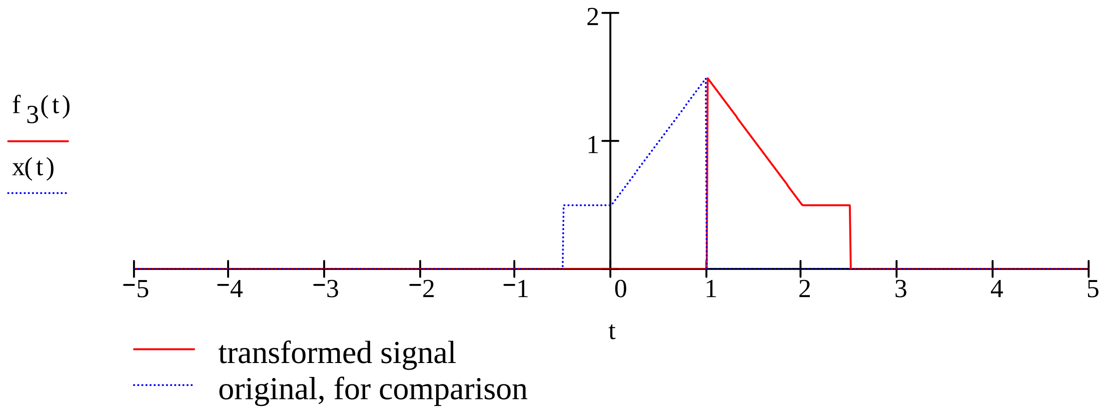


Generally, if you write the transformation in the form $f(t)=y(c\cdot(t-d))$ where *c* and *d* are mnemonic for compression and delay, you'll have no difficulty (delay it by *d* first, then compress the function by *c* about the new centre).

Next, consider combined reflection (negative compression factors) and shift. The function

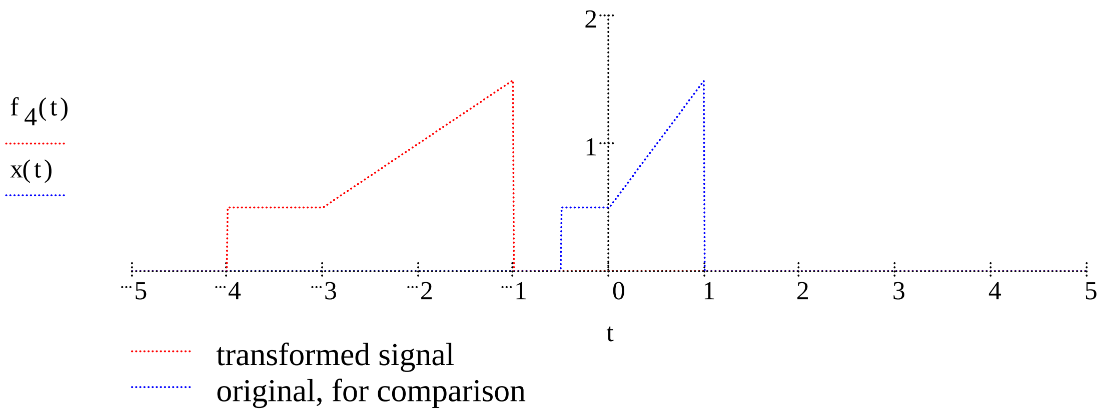
$$f_3(t) := x(-t+2)$$

is not difficult. The argument of *y* is zero at *t*=2, so that's the new location of the centre. The negative sign on *t* means that it is reversed about that centre.



Now, how about this one... $$f_4(t) := x\left[\frac{-1}{6}\cdot(-9-3\cdot t)\right]$$

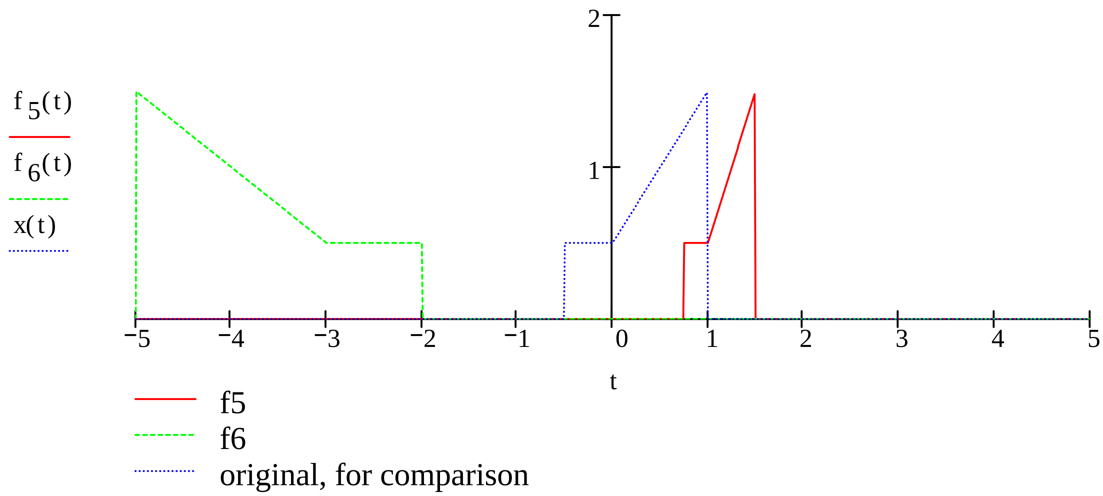
Let's see - compression factor is (-1/6) x (-3) = 1/2, so it's a dilation by 2 with no reflection. The argument is zero at t=-3, so that's where the centre moved.



Here is an area where you can check the effects of parameter choices:

$r := 2$ $\qquad s := -2$ $\qquad c := \frac{-1}{2}$ $\qquad d := -3$

$f_5(t) := x(r \cdot t + s)$ $\qquad f_6(t) := x(c \cdot (t - d))$



5. QUESTIONS

(a) Write $f_4(t)$ in Section 5 above in the form $f_4(t) = x(c(t - d))$ and check the interpretation of *c* as compression and *d* as delay against the graph.

(b) Sketch $y(t)=x(s-t)$ for $s=0$, $s=1$, $s=2$.

(c) Sketch $x\left[2 \cdot \left(\frac{-5}{2} + \frac{3}{2} \cdot t\right)\right]$

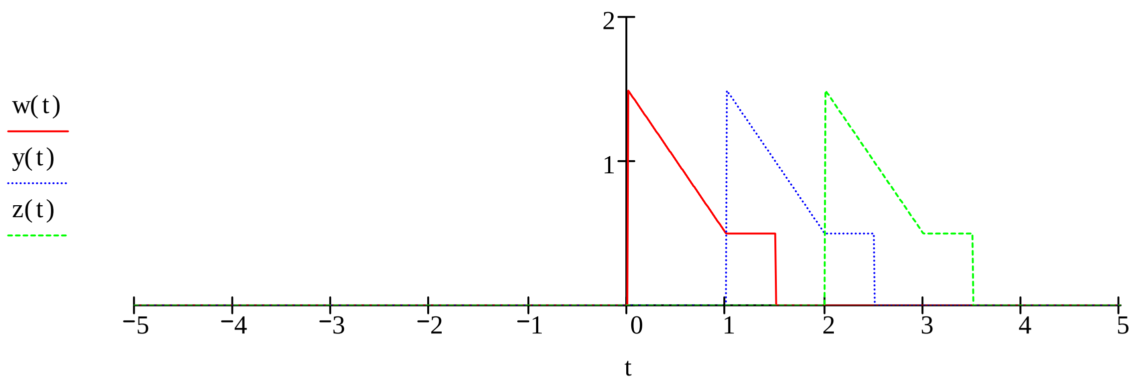
(d) A function can be decomposed to the sum of its even and odd parts as $f(t) = f_e(t) + f_o(t)$ where

$$f_e(t) = \frac{1}{2} \cdot (f(t) + f(-t))$$

What is the corresponding expression for $f_o(t)$?

(e) Finally, here is a form that occurs frequently in convolution. Figure out what these three functions look like. (The solution is at the foot of the page.)

$w(t) := x(1 - t)$ $y(t) := x(2 - t)$ $z(t) := x(3 - t)$



They are just successive time translates of the reversed function $x(-t)$, with the centres at 1, 2 and 3, respectively.