4.3 Comparison of Modulations

- We've seen some of the most common modulation systems. Time to take stock.

  - For 1-D modulations (2-D if complex), such as PAM, QAM, PSK, if we increase the number of bits per symbol without degrading BER, we must rapidly (in fact exponentially) increase the signal energy. The bandwidth does not increase, though.

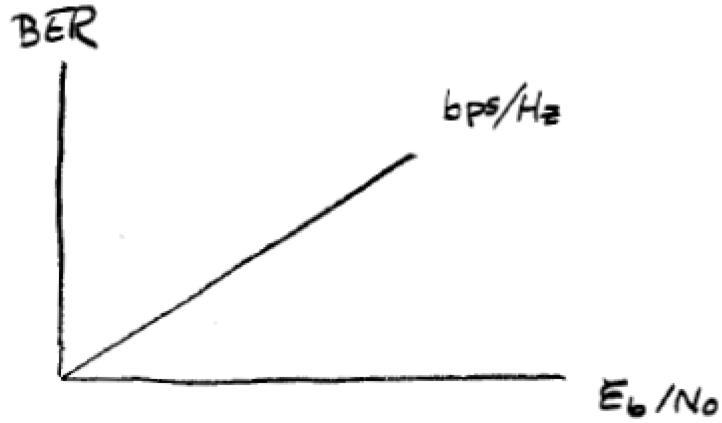
  - For multidimensional modulations like orthogonal, biorthogonal, simplex, increasing the number of bits per symbol to improve BER requires rapidly (exponentially) increasing the bandwidth. No increase in required E_b, though.

  - Multidimensional with use of more than one dimension in a single symbol, like vertices of a hypercube, finite lattice, OFDM, can play it both ways to achieve more throughput: more E_b or more bandwidth.

- It's a three-way tradeoff among

BER	E_b/N_0	bps/Hz
(distance)	power efficiency (energy relative to noise level, or SNR)	spectral efficiency (throughput relative to bandwidth)

Every modulation/coding scheme can be represented as a point in this space.



- Where would we like a modulation to be?
  Are there limits?

- In this section, we'll place our modulations in this space and explore the limits.

4.3.1 Dimensions, Bandwidth and Time

- Part of spectral efficiency is the question of how many dimensions are contained in bandwidth W and time T.

    - How many real numbers are required as coordinates of a signal space with extents W and time T ?

    - Not a very precise question:
        - A signal cannot be strictly timelimited *and* frequency limited.
        - What do we mean by bandwidth (or duration)? Strict limit? 99% energy bandwidth (duration)? 90%? 50% (halfpower bandwidth or time)? RMS bandwidth?

- We'll address it through the Fourier basis, because it's easy. However, more compact basis functions exist: the prolate spheroidal wave functions[1] are strictly limited in one domain and give maximum energy concentration in the other.

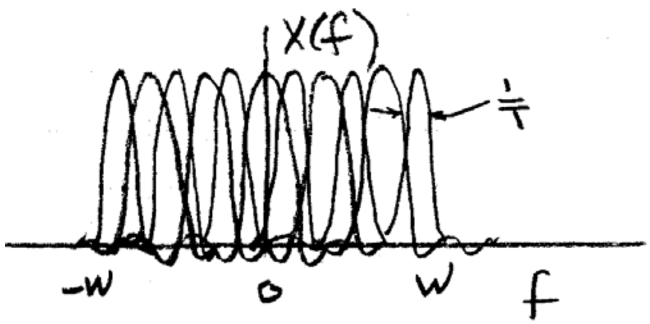
[1] D. Slepian and H.O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty – I," *Bell Syst. Tech. J.*, vol. 40, pp. 43-63, January 1961.

- Consider real, lowpass signals that are strictly time limited to $\left[-T/2, T/2\right]$. Expand such a signal in a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi nt/T} \qquad X(f) = \sum_{n=-\infty}^{\infty} X_n T \operatorname{sinc}\left(fT - n\right)$$

The complex exponentials are separated in frequency by $1/T$ Hz.

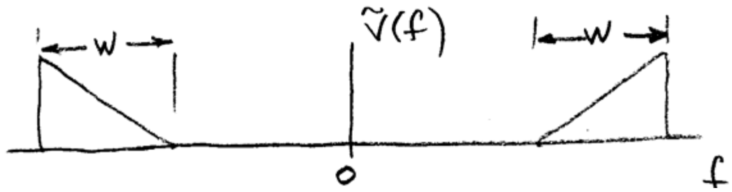
Now suppose it is also approximately bandlimited to W Hz. Then



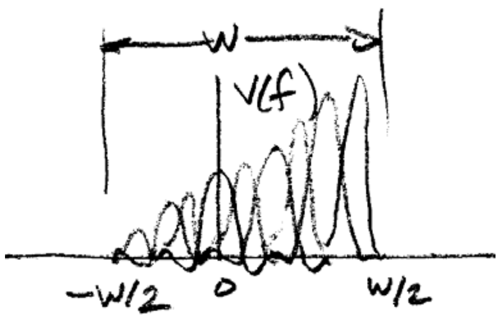
$$x(t) \approx \sum_{n=-WT}^{WT} X_n e^{j2\pi nt/T}$$

Looks like $2WT+1$ basis functions, with all but X_0 complex, for roughly $4WT$ dimensions. However, since $x(t)$ is real, the coefficients are conjugate symmetric $X_{-n} = X_n^*$, so it's roughly $N = 2WT$ real dimensions.

- Now consider real, bandpass signals $\tilde{v}(t)$ with bandwidth *W*, like this one:



It can be represented by the complex baseband signal $v(t)$ with support in $[-W/2, W/2]$. Fourier expansion again:



$$v(t) = \sum_{n=-WT/2}^{WT/2} V_n \, e^{j2\pi nt/T}$$

Now the coefficients need not be conjugate symmetric, so again $N \approx 2WT$ real dimensions.

- Conclude there are $N \approx 2WT$ real dimensions in bandwidth *W* and time *T*. for both lowpass and bandpass signals.

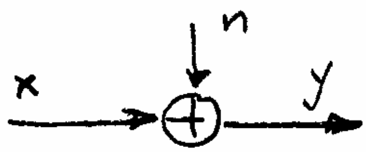
  - Note that real and imaginary components of, say, QAM each count as one dimension.
  - $R_d = N/T = 2W$ dimensions/sec in bandwidth *W*.

4.3.2 Channel Capacity

- We are used to SNR and bit/pulse dictating the BER.

- However, Shannon's capacity theorem sets a surprising limit: if the bps/Hz and SNR are in the right region, communication using long code words becomes essentially error-free. No time to do more than sketch the theory in this course:

  - Suppose a real dimension operates like this:

    Then the "capacity" in bits per dimension is

    $$c = \tfrac{1}{2}\log_2\left(1+\frac{\sigma_x^2}{\sigma_n^2}\right) = \tfrac{1}{2}\log_2\left(1+\frac{E_d}{N_0/2}\right)$$

    where E_d is energy per dimension.

  - Interpretation: a code of length *n* dimensions carrying *k* information bits has a rate $r_d = k/n$ bits/dimension; there is at least one code for which the probability of a codeword error is upper bounded by

    $$P_{ew} \le 2^{-n(c-r_d)};$$

    hence, if we keep $r_d < c$, we can drive the error rate arbitrarily close to zero by increasing the code length *n*.

- The limit on transmission rate is then

$$c = \tfrac{1}{2}\log_2\left(1+\frac{E_d}{N_0/2}\right) \text{ bits/dimension}$$

and using $R_d = 2W$ dim/sec,

$$C = 2Wc = W\log_2\left(1+\frac{E_d}{N_0/2}\right) \text{ bits/sec}$$

- Use bit rate: since $E_d = P/R_d = E_b R_b/R_d = E_b R_b/2W$,

$$C = 2Wc = W\log_2\left(1+\frac{P}{N_0 W}\right) = W\log_2\left(1+\frac{E_b}{N_0}\frac{R_b}{W}\right) \text{ bit/sec}$$

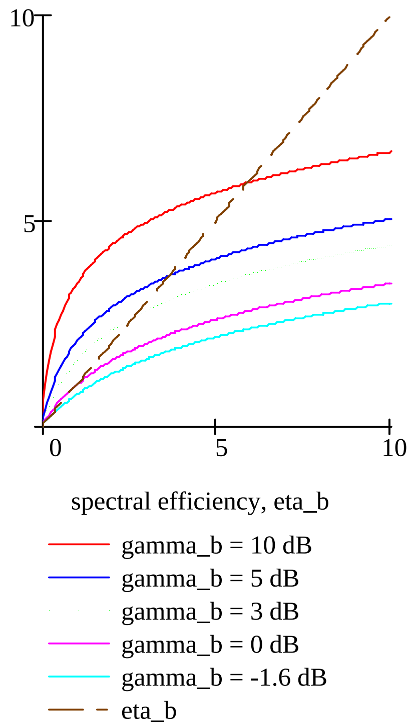
This is the Shannon capacity equation $C = W\log_2(1+SNR)$ bps, expressed in terms of SNR per bit E_b/N_0 and spectral efficiency R_b/W bps/Hz.

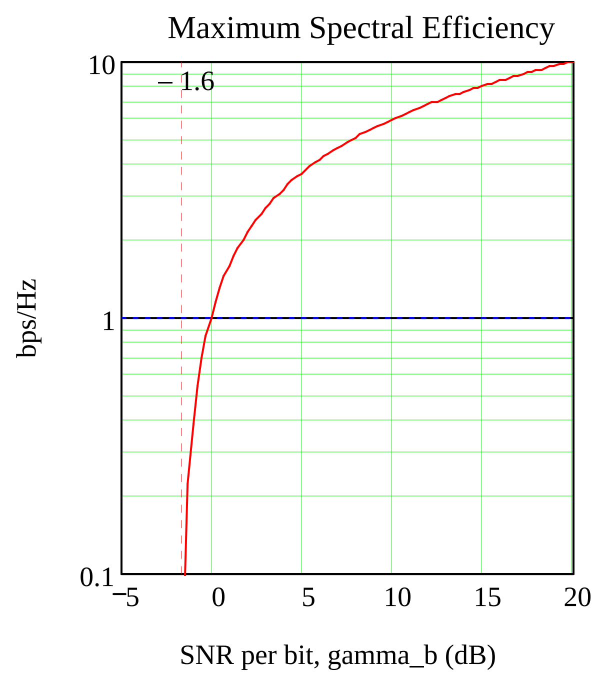
- Optimistically, assume we can reach bit rate $R_b = C$. The we get an implicit equation that sets the limits for SNR and spectral efficiency:

$$\frac{R_b}{W} = \log_2\left(1+\frac{E_b}{N_0}\frac{R_b}{W}\right) \quad \text{or} \quad \eta_b = \log_2(1+\gamma_b\,\eta_b)$$

For any SNR, solution for R_b/W gives the limit of bps/Hz for error-free reception of long codes.

The solution can be viewed as the intersection of left and right sides of the equation (left graph below) and the resulting R_b/W is a function of SNR per bit γ_b (right graph below).





For $\gamma_b < \ln(2)$ (-1.6 dB), there is no solution. Error-free transmission of long codes is not possible.

- One of many implications: Fixed γ_b gives fixed spectral efficiency η_b. Now increase bandwidth W. Throughput $R_b = \eta_b W$ bit/sec goes up in proportion. Good! But it also means that power $P = \gamma_b N_0 R_b$ watt must also be increased in the same proportion.

4.3.3 Comparative Performance of Modulations

- Since Shannon capacity sets the error performance limit in terms of a tradeoff between the dimensionless parameters E_b/N_0 SNR per bit and R_b/W bps/Hz, modulations are compared on a plane with these quantities as axes, and with a specified BER (typically 10^{-5}).

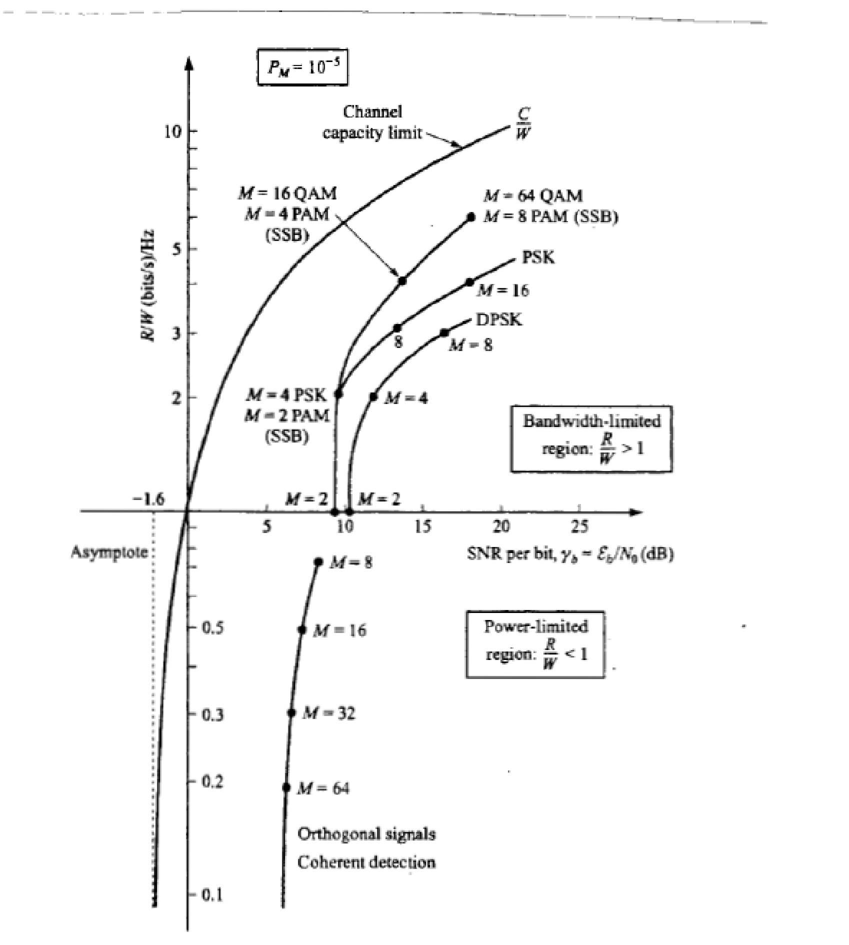


FIGURE 5.2–17
Comparison of several modulation methods at 10^{-5} symbol error probability.

- PAM vs. QAM spectral efficiency: PAM is real, with conjugate symmetric baseband spectrum; *in principle*, could be sent in half the bandwidth using SSB (though difficult in practice for pulses without a null at DC). That's why PAM and QAM have same spectral efficiency in the graph.

4.3.4 Checkpoints

- There is plenty of detail in performances of the various modulation, but some things should be second nature to you:

  - Binary antipodal = 2PAM = 2PSK = BPSK. $P_b = Q\left(\sqrt{2\gamma_b}\right)$. BER is 10^{-3} at just under 7 dB SNR per bit.

  - Binary orthogonal is exactly 3 dB worse than binary antipodal. Hence BER is 10^{-3} at just under 10 dB SNR per bit.

  - 4QAM = QPSK = rotated 4PSK. Their Gray coded BER is exactly the same as binary antipodal for the same γ_b. SER is approximately twice the BER.

  - Independently Gray coded MQAM has exactly the same BER as $\sqrt{M}$-PAM for the same γ_b.

  - If you add another bit to PAM (two more bits to QAM), but keep the error rate of interior points unchanged, you must quadruple the energy per symbol E_s - that is, quadruple the transmit power (for same symbol rate). That's +6 dB. It doesn't quite quadruple E_b, because the new E_s is divided by one more bit (two more bits in QAM).

- For 8PSK and above, adding another bit while keeping error rate constant requires approximately 6 dB more energy per symbol, hence 6 dB more transmit power. Not quite than much increase in energy per bit, because there's an additional bit.

- Adding a bit to an *M*-orthogonal set lowers the error rate without change in γ_b, since the exponential drop in pairwise error probability is faster than the exponential rise in number of neighbours – provided γ_b is more than about 0 dB.